Winter term 2021

Prof. D. Kotschick
S. Gritschacher

Mathematical Gauge Theory II

Sheet 9

Exercise 1. (Dirac operator) Let (X, g) be a closed oriented smooth Riemannian n-manifold with a Spin^c-structure $\mathfrak{s} = (V, \gamma)$ and Spin^c-connection A.

(i) Show that with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ of (TX, g) the Dirac operator $D_A : \Gamma(V) \to \Gamma(V)$ is given by

$$D_A\phi = \sum_{i=1}^{n} \gamma(e_i)\nabla^A_{e_i}\phi\,.$$

(ii) For $\phi, \psi \in \Gamma(V)$ let $\eta \in \Omega^1(X)$ be the 1-form defined by $\eta(Y) = \langle\gamma(Y)\phi, \psi\rangle$ for $Y \in \Gamma(TX)$. Show that at every $p \in X$ the following identity holds:

$$\langle D_A\phi, \psi\rangle_p - \langle\phi, D_A\psi\rangle_p = (\mathrm{d}^*\eta)_p\,.$$

(iii) Conclude that the Dirac operator D_A is formally self-adjoint with respect to the L^2 inner product.

Exercise 2. (Non-negative scalar curvature) Let (X, g) be a smooth closed oriented Riemannian 4-manifold with non-negative scalar curvature endowed with a Spin^c-structure $\mathfrak{s}$. Show, without using the bounds proved in the lecture, that any solution (A, Φ) to the unperturbed Seiberg-Witten equations is reducible, that is, satisfies $\Phi \equiv 0$.

Exercise 3. (Seiberg-Witten equations on $\mathbb{C}\mathrm{P}^2$) Consider $\mathbb{CP}^2$ endowed with the Fubini-Study metric.

(a) Show that the Fubini-Study metric has positive scalar curvature.

(b) Prove that $\mathbb{CP}^2$ is not spin.

(c) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

(d) $\mathbb{CP}^2$ admits a canonical Spin^c-structure by Exercise 4 on Sheet 8. Classify all Spin^c-structures on $\mathbb{C}\mathrm{P}^2$ in terms of cohomology.

Exercise 4. Let X be a smooth oriented Riemannian 4-manifold equipped with a compatible almost complex structure J, and let $\mathfrak{s}_{\mathrm{can}}$ be the canonical Spin^c-structure (see Exercise 4, Sheet 8).

(i) Prove that $p_1(X) = c_1(X)^2 - 2c_2(X)$.

(ii) Show that $\langle c_1(L_{\mathfrak{s}_{\mathrm{can}}})^2, [X]\rangle = 2\chi(X) + 3\sigma(X)$.

[You may use the fact that $\chi(X) = \langle c_2(X), [X]\rangle$.]

You can hand in solutions in the lecture on Thursday, 20 January 2022.