Winter term 2021

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Mathematical Gauge Theory II

Sheet 8

Exercise 1. (Harmonic maps) Let (X, g) be a compact oriented smooth Riemannian manifold and let $u : X \to S^1$ be a smooth map. Show that u is harmonic, that is $\mathrm{d}^*(u\,\mathrm{d}\,u^{-1}) = 0$, if and only if u is a critical point of the energy functional

$$\mathcal{E}(u) := \frac{1}{2}\int_X \|\mathrm{d}\,u\|^2 \;\mathrm{d}\,vol_g\,.$$

Exercise 2. Show that the maps

$$\mathfrak{e}\colon\; \mathcal{G}^\perp \times \mathcal{S} \longrightarrow \mathcal{A}$$
$$(e^{if}, (A_0 + a, \Phi)) \longmapsto (A_0 + a - i\,\mathrm{d}\,f, e^{if}\Phi)$$

and

$$\mathfrak{e}^{-1}\colon\; \mathcal{A} \longrightarrow \mathcal{G}^\perp \times \mathcal{S}$$
$$(A_0 + b, \Psi) \longmapsto (e^{-G(\mathrm{d}^* b)}, A_0 + b - \mathrm{d}(G(\mathrm{d}^* b)), e^{G(\mathrm{d}^* b)}\Psi))$$

are inverse to each other.

Exercise 3. Let (H, g) be a real $2n$-dimensional Euclidean inner product space, and $J : H \to H$ a complex structure compatible with the inner product, that is, an orthogonal transformation satisfying $J^2 = -id$. Equip $V := H \otimes_{\mathbb{R}} \mathbb{C}$ with the Hermitian inner product

$$h(v \otimes \lambda, w \otimes \mu) := \lambda\bar{\mu}\, g(v, w)$$

for $v, w \in H$ and $\lambda, \mu \in \mathbb{C}$.

1. Show that there is an orthogonal decomposition $V = V^{1,0} \oplus V^{0,1}$ into the $\pm i$-eigenspaces of the $\mathbb{C}$-linear extension of J. Conclude that there is an induced decomposition $\bigwedge^k V = \bigoplus_{p+q=k} \bigwedge^{p,q} V$ where $\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$.
2. Show that the $\mathbb{C}$-linear extension of the Hodge star $*\colon \bigwedge^k V \to \bigwedge^{2n-k} V$ satisfies $\alpha \wedge *\bar{\beta} = h(\alpha, \beta)\,\mathrm{vol}$ and conclude that $*$ maps $\bigwedge^{p,q} V$ to $\bigwedge^{n-q,n-p} V$.

(please turn)

Exercise 4. Let H be a real 4-dimensional Euclidean inner product space, and $J : H \to H$ a complex structure compatible with the inner product. Let $V = H \otimes_{\mathbb{R}} \mathbb{C}$ and for $v \in V$ let $v^{1,0} := (v - iJ(v))/2$ and $v^{0,1} := (v + iJ(v))/2$ denote the projection of v to $V^{1,0}$ and $V^{0,1}$, respectively. Set

$$V_+ := V^{0,0} \oplus V^{0,2}\,, \qquad V_- := V^{0,1}\,.$$

1. Show that $\bigwedge^* V^{0,1} = V_+ \oplus V_-$ becomes a Clifford module by defining Clifford multiplication on $(\alpha, \beta) \in V_+$ by
$$\gamma(v)(\alpha, \beta) = \sqrt{2}(v^{0,1} \wedge \alpha - *(v^{1,0} \wedge *\beta))\,,$$
and on $\psi \in V_-$ by
$$\gamma(v)\psi = \sqrt{2}(- * (v^{1,0} \wedge *\psi), v^{0,1} \wedge \psi)\,.$$
[It may be helpful to recall that $*^2 = (-1)^{k(n-k)}$ on $\bigwedge^k V$ where $n = \dim_{\mathbb{C}}(V)$.]

2. Let X be a smooth oriented Riemannian 4-manifold equipped with a compatible almost complex structure $J : TX \to TX$, that is, J is a fibrewise complex structure compatible with the metric. Conclude from (1) that X has a canonical $Spin^c$-structure, and determine its characteristic line bundle.

You can hand in solutions in the lecture on Thursday, 13 January 2021.