Winter term 2021

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# Mathematical Gauge Theory II

Sheet 6

**Exercise 1.** Let $H \to X$ be a real Euclidean vector bundle of rank four equipped with a $\mathrm{Spin}^c$-structure, which we regard as a principal $\mathrm{Spin}^c(4)$-bundle $Q \to X$. Prove that the characteristic line bundle of the $\mathrm{Spin}^c$-structure is the line bundle associated to $Q$ via the homomorphism

$$\begin{aligned}\mathrm{Spin}^c(4) \cong (\mathrm{Spin}(4) \times S^1)/(\mathbb{Z}/2) &\longrightarrow S^1 \\ [\alpha, \beta] &\longmapsto \beta^2 .\end{aligned}$$

**Exercise 2.** (Hodge dual and differential of 1-forms) Let $(X^n, g)$ be an oriented Riemannian manifold with Levi-Civita connection $\nabla$ and $\eta \in \Omega^1(X)$ a 1-form. Let $p \in X$ and $e_1, \dots, e_n$ an oriented local frame for $TX$ on an open neighbourhood around $p$ such that

$$(\nabla e_i)(p) = 0 \quad \forall i \in \{1, \dots, n\}.$$

Prove that in $p$

$$* d * \eta = \sum_{i=1}^{n} L_{e_i} \eta(e_i),$$

where $*$ denotes the Hodge star.

**Exercise 3.**

1. Prove that if $P\colon H_1 \longrightarrow H_2$ is a Fredholm operator between Hilbert spaces, then $\mathrm{coker} P \cong \ker P^*$.
2. Let $X$ be a closed oriented connected Riemannian manifold and let $\mathrm{d}^*$ be the formal adjoint of $\mathrm{d}$ with respect to the $L^2$ scalar product. Show that the operator

$$P = \mathrm{d} + \mathrm{d}^* \colon \bigoplus_{k \text{ even}} \Omega^k(X) \longrightarrow \bigoplus_{k \text{ odd}} \Omega^k(X)$$

is Fredholm by showing that $\dim \ker(P)$ and $\dim \mathrm{coker}(P)$ are finite. Compute $\mathrm{ind}(P)$.

(please turn)

**Exercise 4.** (Spinor identities) Let $\Gamma(V_+)$ be the space of positive spinors associated with a Spin$^c$-structure on a closed oriented Riemannian 4-manifold. We give $\mathrm{End}(V_+)$ the inner product induced by

$$\langle A, B\rangle = \operatorname{tr}\left(AB^\dagger\right)$$

and define

$$\begin{aligned} \sigma\colon \Gamma(V_+) &\longrightarrow \Omega^2_+(X, i\mathbb{R}) \\ \Phi &\longmapsto \sigma(\Phi,\Phi) = \gamma^{-1}\left((\Phi\otimes\Phi^\dagger)_0\right) \end{aligned}$$

as in the lecture. Prove that for all $\omega, \eta \in \Omega^2_+(X, i\mathbb{R})$ and $\Phi \in \Gamma(V_+)$ the following identities hold:

$$\begin{aligned} \langle\gamma(\omega), \gamma(\eta)\rangle &= 4\langle\omega, \eta\rangle \\ \langle\gamma(\omega)\Phi, \Phi\rangle &= 4\langle\omega, \sigma(\Phi,\Phi)\rangle \\ |\Phi|^4 &= 8|\sigma(\Phi,\Phi)|^2. \end{aligned}$$

You can hand in solutions in the lecture on Thursday, 2 December 2021.