Winter term 2021

Prof. D. Kotschick
S. Gritschacher

Mathematical Gauge Theory II

Sheet 5

Exercise 1. (Invariants of 4-manifolds) Let M and N be two closed simply connected oriented smooth 4-manifolds.

1. Prove that M and N are homeomorphic if and only if the the following invariants agree:
   - Euler characteristic χ
   - signature σ
   - parity (even or odd) of the intersection form.
2. Identify $M\#\overline{\mathbb{CP}}^2$ up to homeomorphism.
3. Assume $\sigma(M) = -\sigma(N)$ and even intersection forms Q_M, Q_N. Find a 4-manifold homeomorphic to $M\#N$.

Exercise 2. (Complete intersections) Let $d = (d_1, d_2, \ldots, d_r)$ be an r-tuple of natural numbers and consider the intersection of r smooth hypersurfaces X_{d_i} of degree d_i in $\mathbb{CP}^{r+2}$:

$$S_d = X_{d_1} \cap X_{d_2} \ldots \cap X_{d_r}.$$

We assume that for all $k = 2, \ldots, r$ the hypersurface X_{d_k} intersects $X_{d_1} \cap \ldots \cap X_{d_{k-1}}$ transversely. Then S_d is a smooth complex surface, called a *complete intersection of multidegree d*.

1. Suppose submanifolds M and N of a manifold W intersect transversely. Show that the normal bundles in W are related by $\nu(M \cap N) = \nu(M)|_{M\cap N} \oplus \nu(N)|_{M\cap N}$.
2. Calculate the Chern classes $c_1(S_d)$ and $c_2(S_d)$.
3. Determine those multidegrees d for which S_d is a $K3$ surface.

(please turn)

Exercise 3. (Chern classes of tensor products)

1. Let V, W be complex vector bundles of rank 2. Use the splitting principle to prove that

$$c_1(V \otimes W) = 2\left(c_1(V) + c_1(W)\right)$$
$$c_2(V \otimes W) = 2\left(c_2(V) + c_2(W)\right) + c_1^2(V) + c_1^2(W) + 3c_1(V)c_1(W).$$

2. Let V_+ be the spinor bundle of a Spin^c-structure over an oriented Riemannian 4-manifold. Use an isomorphism induced from Clifford multiplication to prove that

$$p_1\left(\Lambda^2_+\right) = c_1^2\left(V_+\right) - 4c_2\left(V_+\right).$$

Exercise 4. (S^2-bundles over Σ_g) Let Σ_g denote the surface of genus g.

1. Suppose $D^2 \subset \Sigma_g$ is a small disk around a point. Show that $\Sigma_g \setminus D^2$ is homotopy equivalent to a 1-point union $\bigvee_{i=1}^{2g} S_i^1$ of $2g$ circles.

2. Prove that for every $g \geq 0$ there are at most two orientable S^2-bundles over Σ_g up to diffeomorphism.

You can hand in solutions in the lecture on Thursday, 25 November 2021.