Winter term 2021

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Mathematical Gauge Theory II

Sheet 1

Exercise 1. (The standard Clifford module of $\mathbb{R}^4$) Let e_0, e_1, e_2, e_3 be an orthonormal basis of $\mathbb{R}^4$ with the standard Euclidean scalar product. Define a linear map $\gamma\colon \mathbb{R}^4 \to \mathrm{End}(\mathbb{C}^4)$ by

$$\gamma(e_j) = A_j = \begin{pmatrix} 0 & -B_j^\dagger \\ B_j & 0 \end{pmatrix}$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Prove that $(\mathbb{C}^4, \gamma)$ with the standard Hermitian scalar product is a Clifford module for $(\mathbb{R}^4, g_{\mathrm{can}})$.

Exercise 2. (Clifford multiplication with forms) Let $(\mathbb{C}^4, \gamma)$ be the standard Clifford module for $(\mathbb{R}^4, g_{\mathrm{can}})$ from Exercise 1. Prove that:

1.
$$\gamma(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

2. Under γ the self-dual two-forms

$$e_0 \wedge e_1 + e_2 \wedge e_3$$
$$e_0 \wedge e_2 - e_1 \wedge e_3$$
$$e_0 \wedge e_3 + e_1 \wedge e_2$$

   act non-trivially on $\mathbb{C}^2_+$ as $2B_1, 2B_2$ and $2B_3$, respectively, and are zero on $\mathbb{C}^2_-$.

3. The map γ induces isomorphisms

$$\big(\Lambda^1(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)\big) \otimes \mathbb{C} \cong \mathrm{Hom}(\mathbb{C}^2_+, \mathbb{C}^2_-) \oplus \mathrm{Hom}(\mathbb{C}^2_-, \mathbb{C}^2_+)$$
$$\Lambda^2_\pm(\mathbb{R}^4) \otimes \mathbb{C} \cong \mathrm{End}_0(\mathbb{C}^2_\pm)$$
$$\Lambda^4(\mathbb{R}^4) \otimes \mathbb{C} \cong \mathbb{C} \cdot \mathrm{Id}_{\mathbb{C}^2_\pm},$$

   where End_0 denotes the trace-free endomorphisms.

(please turn)

Exercise 3. (Schur's Lemma) Let (V, γ) be an irreducible Clifford module. Prove that every automorphism of (V, γ), i.e. every isomorphism $f\colon V \to V$ of Clifford modules, is of the form $f(\phi) = \lambda\phi$ for some constant $\lambda \in S^1$.

Exercise 4. (Anti-linear automorphisms I) Let $J\colon V \to V$ be a complex anti-linear automorphism of a standard (i.e. irreducible) Clifford module. Show that $J^2 = \pm\mathrm{Id}_V$.

You can hand in solutions in the lecture on Thursday, 21 October 2021.