Ex. 3. Claim: $A_i$ commute with $\nabla$.

Proof: Step 1 of the hint: We use the Koszul formula to expand the LHS. There are two contributions:

$$2g(\nabla_X JY, Z) = X(g(JY, Z)) + JY\, g(X, Z) - Z\, g(X, JY)$$
$$- g([JY, X], Z) - g([X, Z], JY) - g([JY, Z], X)$$

$$2g(J\nabla_X Y, Z) = 2g(\nabla_X Y, JZ)$$

g is J-invariant

$$= X(g(Y, JZ)) + Y\, g(X, JZ) - JZ\, g(X, Y)$$
$$- g([Y, X], JZ) - g([X, JZ], Y) - g([Y, JZ], X)$$

We expand the RHS using the formula for exterior derivative:

$$d\omega(X, Y, Z) = X(\omega(Y, Z)) - Y\,\omega(X, Z) + Z\,\omega(X, Y)$$
$$- \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X)$$
$$= - X\, g(Y, JZ) + Y\, g(X, JZ) - Z\, g(X, JY)$$

def of $g(X,Y) = \omega(X, JY)$

$$+ g([X, Y], JZ) - g([X, Z], JY) + g([Y, Z], JX)$$

$$- d\omega(X, JY, JZ) = - X\, g(JY, Z) + JY\, g(X, Z) - JZ\, g(X, Y)$$
$$+ g([X, JY], Z) - g([X, JZ], Y) - g([JY, JZ], JX)$$

Now we match up both sides and see what's left:

that should be precisely $- g(N(Y, Z), JX)$. (do in class).

## Step 2 of the hint

If the $\pm i$-eigenbundle of $J$ are closed under $[\cdot,\cdot]$, then a computation similar to the one in the lecture (for an almost product str.) shows that $N \equiv 0$.

Once we pass from $TM$ to $TM \otimes \mathbb{C}$, any $X \in \Gamma(TM)$ can be written $X = X_+ + X_-$ with $X_\pm$ a section of the $\pm i$-eigenbundle of $J$ (which we called $T^{1,0}$, $T^{0,1}$ in the lecture)

Then: 
$$N(Y_+, Z_+) = [Y_+, Z_+] + J\underbrace{[JY_+, Z_+]}_{i\underbrace{[Y_+, Z_+]}_{\in\, i\text{-eigenspace}}} + J\underbrace{[Y_+, JZ_+]}_{i[Y_+, Z_+]} - \underbrace{[JY_+, JZ_+]}_{i^2[Y_+, Z_+]}$$

$$= 2[Y_+, Z_+] + 2iJ\underbrace{[Y_+, Z_+]}_{i\text{-eigenbundle}} = 0.$$

Similarly for $N(Y_-, Z_-) = 0$.

On the other hand,

$$N(Y_+, Z_-) = [Y_+, Z_-] + J([JY_+, Z_-]) + J[Y_+, JZ_-] - [JY_+, JZ_-]$$

$$= [Y_+, Z_-] + J(i[Y_+, Z_-] - i[Y_+, Z_-]) - [Y_+, Z_-] = 0.$$

Similarly, $N(Y_-, Z_+) = 0$.

$\Rightarrow N \equiv 0$.

Step 3 of the hint

$\omega_i$ symplectic $\Rightarrow$ $d\omega_i = 0$.

lecture showed that the $\pm i$ - eigenbundle of $A_i$ are closed under $[\cdot,\cdot]$, so by step 2 $N \equiv 0$.

by step 1, $g((\nabla_X J)Y, Z) = 0 \quad \forall X, Y, Z$.

by non-degeneracy of $g$, $\nabla J \equiv 0$. $\square$

Claim: $\omega_i$ is $\nabla$- compatible.

Proof: $L_X \omega(Y,Z) = - L_X g(JY, Z)$

$$\underset{\nabla \text{ is } g\text{-compatible}}{=} - g(\nabla_X JY, Z) - g(JY, \nabla_X Z)$$

$$\underset{\nabla J = 0}{=} - g(J\nabla_X Y, Z) - g(JY, \nabla_X Z)$$

$$\underset{\text{def of } \omega}{=} \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z).$$

(here $\omega := \omega_i$ and $J := A_i$). $\square$

Ex. 4. $\mathfrak{nil}_3 \times \mathbb{R}$ spanned by $e_1, \dots, e_4$ with $[e_1, e_2] = e_3$

let $\alpha_1, \dots, \alpha_4$ be the dual basis ( $\alpha_i(e_j) = \delta_{ij}$ ) and

extend them to left-inv. 1-forms on $Nil_3 \times \mathbb{R}$

just means $l_g^* \alpha = \alpha \quad \forall g \in Nil_3 \times \mathbb{R}$

$l_g$ = left-multiplication by $g$.

is the group of upper uni-triangular matrices $\left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \middle| \, x, y, t \in \mathbb{R} \right\}$

was not part of the exercise

formula for exterior derivative says

$$d\alpha_i(X, Y) = X(\alpha_i(Y)) - Y(\alpha_i(X)) - \alpha_i([X, Y])$$

mistake here last time where I forgot $L_X, L_Y$ ⇝ correct in class

made

let $X_1, \dots, X_4$ be the left-inv. vector fields associated to $e_1, \dots, e_4$.

Evaluating $d\alpha$ on $X_1, \dots, X_4$ we get:

$$d\alpha_i(X_j, X_k) = X_j \, \alpha_i(X_k) - X_k \, \alpha_i(X_j) - \alpha_i([X_j, X_k])$$

now $\alpha_i(X_k)(g) = \alpha_i(dl_g(e_k)) = (\underbrace{l_g^* \alpha_i}_{\text{dual to } e_i})(e_k) = \delta_{ik}$

is constant, so $X_j(\alpha_i(X_k)) = 0$. Similarly $X_k \, \alpha_i(X_j) = 0$.

$$\Rightarrow \quad d\alpha_i(X_j, X_k) = -\underbrace{\alpha_i}_{\text{left-inv. form}}([X_j, X_k]) = -\underbrace{\alpha_i}_{\text{covector dual to } e_i}([e_j, e_k]).$$

From the relations in $\mathfrak{nil}_3 \times \mathbb{R}$ we thus find

$$d\alpha_3 = -\alpha_1 \wedge \alpha_2 = \alpha_2 \wedge \alpha_1$$

and $d\alpha_i = 0 \quad \forall i \neq 3$.

$\alpha, \beta, \gamma$ are closed:

$$\begin{aligned} d\alpha &= d(\alpha_3 \wedge \alpha_2) + d(\alpha_1 \wedge \alpha_4) \\ &= d\alpha_3 \wedge \alpha_2 - \alpha_3 \wedge \cancel{d\alpha_2}^{0} + \cancel{d\alpha_1}^{0} \wedge \alpha_4 - \alpha_1 \wedge \cancel{d\alpha_4}^{0} \\ &= \alpha_2 \wedge \alpha_1 \wedge \alpha_2 = 0. \end{aligned}$$

Similarly for $\beta$ and $\gamma$.

$\alpha, \beta, \gamma$ are non-degenerate

$$\alpha \wedge \alpha = -2\, \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \neq 0 \quad \text{nowhere zero.}$$

(at every point of $Nil_3 \times \mathbb{R}$ $\alpha_1, \dots, \alpha_4$ are covectors forming a basis for the cotangent space, so $\alpha_1 \wedge \dots \wedge \alpha_4$ is nowhere zero)

$\alpha, \beta, \gamma$ are left-invariant

since pullback commute with $\wedge$.

$\alpha, \beta, \gamma$ define a hypersymplectic structure for this we must calculate the recursion operators:

$$\begin{tikzcd} & \gamma \arrow[dl, "A"'] \arrow[dr, "B"] & \\ \alpha \arrow[rr, "J"'] & & \beta \end{tikzcd}$$

E.g. find $A$ s.t. $\gamma(A-,-) \overset{!}{=} \alpha$.

$$\Rightarrow \left.\begin{aligned} Ae_1 &= e_2 \\ Ae_2 &= e_1 \\ Ae_3 &= -e_4 \\ Ae_4 &= -e_3 \end{aligned}\right\} \Rightarrow A = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$$

$$\gamma(B-,-) \overset{!}{=} \beta \Rightarrow \left.\begin{aligned} Be_1 &= e_1 \\ Be_2 &= -e_2 \\ Be_3 &= e_3 \\ Be_4 &= -e_4 \end{aligned}\right\} \Rightarrow B = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$\cancel{\gamma}\ \alpha(J-,-) \overset{!}{=} \beta \Rightarrow \left.\begin{aligned} Je_1 &= e_2 \\ Je_2 &= -e_1 \\ Je_3 &= -e_4 \\ Je_4 &= e_3 \end{aligned}\right\} \Rightarrow J = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$$

We see that $A^2 = B^2 = id = -J^2$. $\square$