Ex. 1. "$\Rightarrow$" Suppose the action is Hamiltonian

Want to show $\{\mu^X, \mu^Y\} = \mu^{[X,Y]}$.

$$\{\mu^X, \mu^Y\}(p) \overset{\text{Ex. 2 Sheet 5}}{=} \omega(X^\#, Y^\#)(p)$$

$$= (i_{X^\#}\omega)(Y^\#)(p)$$

$$\overset{\mu^X \text{ Hamilt. fct}}{=} (d\mu^X)(Y^\#)(p)$$

$$= Y^\#(\mu^X)(p)$$

$$\overset{\text{Def. of } Y^\#}{=} \frac{d}{dt}\Big|_0 \mu^X(\psi_{\exp(tY)}(p))$$

Action is Hamiltonian

$$= \frac{d}{dt}\Big|_0 \underbrace{\mu^{Ad_{\exp(tY)^{-1}} X}(p)}_{= \langle \mu(p), Ad_{\exp(-tY)} X \rangle}$$

note that $\langle \mu(p), - \rangle : \mathfrak{g} \to \mathbb{R}$ is a linear map, so it commutes with the t-derivative (chain rule)

$$= \langle \mu(p), \frac{d}{dt}\Big|_0 Ad_{\exp(-tY)} X \rangle$$

Def. of commutator via adjoint repr.:

$[X,Y] = ad_X(Y) = \frac{d}{dt}\Big|_0 Ad_{\exp(tX)}(Y)$

$$= \langle \mu(p), [-Y, X] \rangle$$

$$= \mu^{[X,Y]}(p) \qquad \square$$

"$\Leftarrow$" Suppose $X \mapsto \mu^X$ is a Lie algebra hom., so $\mu^{[X,Y]} = \{\mu^X, \mu^Y\}$. Must check $\mu(\psi_g(p)) = Ad_g^*(\mu(p))$ for all $g \in G$.

Let's check this when $g = \exp(tX)$, $X \in \mathfrak{g}$

Notice that $\mu(\varphi_{\exp(tX)}(p))$ and $Ad^*_{\exp(tX)}(\mu(p))$

agree for small values of t if their derivative

at $t=0$ agree.

$$\forall Y \in \mathfrak{g}: \quad \left\langle \frac{d}{dt}\Big|_0 \mu(\varphi_{\exp(tX)}(p)), Y \right\rangle$$

$\langle -, Y\rangle$ linear map $\mathfrak{g}^* \to \mathbb{R}$ + chain rule

$$= \frac{d}{dt}\Big|_{t=0} \langle \mu(\varphi_{\exp(tX)}(p)), Y\rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \mu^Y(\varphi_{\exp(tX)}(p))$$

$$= X^\#(\mu^Y(p))$$

$\{\mu^Y, \mu^X\} = \omega(Y^\#, X^\#)$
$= i_{X^\#} i_{Y^\#} \omega$
$= i_{X^\#} d\mu^Y = X^\#(\mu^Y)$

$$= \{\mu^Y, \mu^X\}(p)$$

$X \mapsto \mu^X$ is Lie alg. hom.

$$= \mu^{[Y,X]}(p)$$

$$= \langle \mu(p), [-X, Y]\rangle$$

as on p. 1.

$$= \frac{d}{dt}\Big|_0 \langle \mu(p), Ad_{\exp(-tX)}(Y)\rangle$$

def. of dual repr.

$$= \frac{d}{dt}\Big|_0 \langle Ad^*_{\exp(tX)} \mu(p), Y\rangle$$

$\langle -, Y\rangle$ linear + chain rule

$$= \left\langle \frac{d}{dt}\Big|_{t=0} Ad^*_{\exp(tX)} \mu(p), Y \right\rangle$$

$$\Rightarrow \frac{d}{dt}\Big|_0 \mu(\varphi_{\exp(tX)}(p)) = \frac{d}{dt}\Big|_{t=0} Ad^*_{\exp(tX)} \mu(p) \Rightarrow \mu(\varphi_{\exp(tX)}(p)) = Ad^*_{\exp(tX)} \mu(p)$$

for all X and small t.

But $\exp : \mathfrak{g} \to G$ is a local diffeomorphism around $0 \in \mathfrak{g}$ so the statement holds for all $g \in G$ in a neighbourhood of $1 \in G$. Such a neighbourhood generates $G \Rightarrow$ equivariance holds for all $g \in G$.

$\square$

Ex. 2. Here are a few things one may want to check first:

- $\{\cdot,\cdot\}$ is a Lie bracket on $C^\infty(M)$:

bilinear ✓

antisymmetric ✓

Jacobi-identity: $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$

$$= \omega(X_{\{f,g\}}, X_h) + \omega(X_{\{g,h\}}, X_f) + \omega(X_{\{h,f\}}, X_g)$$

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$$= \omega([X_g, X_f], X_h) + \omega([X_h, X_g], X_f) + \omega([X_f, X_h], X_g)$$

formula for ext. derivative

$$= \underbrace{d\omega}_{=0}(X_f, X_g, X_h) + \underbrace{(\mathcal{L}_{X_f}\omega)}_{=0}(X_g, X_h) - \underbrace{(\mathcal{L}_{X_g}\omega)}_{=0}(X_f, X_h)$$

$$+ \underbrace{(\mathcal{L}_{X_h}\omega)}_{=0}(X_f, X_g)$$

$$\left(\text{since } \mathcal{L}_{X_h}\omega = i_{X_h}\underbrace{d\omega}_{=0} + d\underbrace{i_{X_h}\omega}_{dh} = ddh = 0\right).$$

$$= 0.$$

- $[X, Y]^\# = [Y^\#, X^\#]$

$$[Y^\#, X^\#](p) = \frac{d}{dt}\Big|_0 d\psi_{\exp(-tY)} \underbrace{X^\#(\psi_{\exp(tY)}(p))}$$

I mean the value of $X^\#$ at the point $\psi_{\exp(tY)}(p)$

def. of $X^\#$

$$= \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \underbrace{\psi_{\exp(-tY)}(\psi_{\exp(sX)}(\psi_{\exp(tY)}(p)))}$$

$$= \psi_{\exp(-tY)\exp(sX)\exp(tY)} = \psi_{\exp(s\,\mathrm{Ad}_{\exp(-tY)}X)}$$

$$= \quad [-Y, X]^{\#}(p) \quad = \quad [X,Y]^{\#}(p).$$

recall $G \times M \xrightarrow{\psi} M$

$(d_{(1,p)}\psi)(X,0) = X^{\#}(p)$ (we did this in one of the EX. classes)

now $\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \psi_{\exp(s\,Ad_{\exp(-tY)}X)} = (d\psi)\left(\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \exp(s\,Ad_{\exp(-tY)}X), 0\right)$

$$= (d\psi)\left(\frac{d}{dt}\Big|_0 Ad_{\exp(-tY)}X, 0\right)$$

$$= d\psi([-Y,X], 0)$$

$$= [-Y,X]^{\#}.$$

Now let's do the exercise: we'll have to assume that

<u>M is connected</u>.

(1) $\{\mu^X, \mu^Y\}$ is a Hamiltonian function for $X_{\{\mu^X,\mu^Y\}} \overset{\text{Sheet 5}}{=} [X_{\mu^Y}, X_{\mu^X}]$

$$= [Y^{\#}, X^{\#}]$$

$$= [X,Y]^{\#}.$$

See above

But $\mu^{[X,Y]}$ is also Hamiltonian function for $[X,Y]^{\#}$.

$\Rightarrow d(\mu^{[X,Y]} - \{\mu^X,\mu^Y\}) = 0$

$\underset{M \text{ connected}}{\Rightarrow} \{\mu^X,\mu^Y\} - \mu^{[X,Y]} = \text{constant} =: \tau(X,Y).$

Clearly, $\tau(X,Y)$ is bilinear, because $X \mapsto \mu^X$ is assumed linear.

To verify the identity for τ we use:

$$\{\mu^{[X,Y]}, \mu^Z\} = \{\mu^{[X,Y]} + \underbrace{\tau(X,Y)}_{\text{constant, so } \{\cdot,\cdot\} \text{ will be } 0}, \mu^Z\}$$

$$= \{\{\mu^X, \mu^Y\}, \mu^Z\}$$

now both $[\cdot,\cdot]$ and $\{\cdot,\cdot\}$ satisfy the Jacobi identity and this implies the identity for τ.

(2) Define $\tilde{\mu}^X := \mu^X + \sigma(X)$

Then
$$\tilde{\mu}^{[X,Y]} = \mu^{[X,Y]} + \sigma([X,Y])$$

$$= \{\mu^X, \mu^Y\} \underbrace{- \tau(X,Y) + \sigma([X,Y])}_{=0}$$

$$= \{\mu^X, \mu^Y\}$$

$$= \{\mu^X + \underbrace{\sigma(X)}_{\text{const.}}, \mu^Y + \underbrace{\sigma(Y)}_{\text{const.}}\}$$

$$= \{\tilde{\mu}^X, \tilde{\mu}^Y\}$$

$\Rightarrow$ $\tilde{\mu} : M \to \mathfrak{g}^*$ is a moment map by Exercise 1

(since $X \mapsto \tilde{\mu}^X$ is a Lie algebra hom.)

$\Rightarrow$ the action is Hamiltonian.

Rmk the identity for τ should be interpreted as a cocycle condition: it says that $\tau : \Lambda^2 \mathfrak{g} \to \mathbb{R}$ is a 2-cocycle in the Lie algebra cohomology of $\mathfrak{g}$. The assumption that $\tau(X,Y) = \sigma([X,Y])$ says that τ is a coboundary.