Let's look more generally at a vector bundle $\pi\colon E \to X$. Then $d\pi\colon TE \to TX$ is a map of vector bundles covering π, and by the universal property of the pullback we get a map $\pi_*\colon TE \to \pi^*TX$ of vector bundles over E which is $d\pi$ in every fibre. In particular, it is surjective in every fibre because π is a submersion, so $\ker(\pi_*)$ is a subbundle of TE. It consists precisely of the tangent vectors tangent to the fibres of $\pi\colon E \to X$. Moreover,

$$\begin{aligned} T^{\text{vert}}E := \ker(\pi_*) &\cong \{(x,v,w) \mid (x,v) \in E,\, w \in T_vE_x \cong E_x\} \\ &\cong \{(x,v,w) \mid x \in X,\, v,w \in E_x\} \\ &= \pi^*E\,. \end{aligned}$$

We have a short exact sequence

$$0 \to T^{\text{vert}}E \xrightarrow{\text{incl}} TE \xrightarrow{\pi_*} \pi^*TX \to 0\,.$$

The sequence splits by giving a section $s\colon \pi^*TX \to TE$ for π_*. Since s is fibrewise injective, that's equivalent to specifying a subbundle of $V \subset TE$ which is everywhere complementary to $T^{\text{vert}}E \subset TE$, because for such a bundle $V \xrightarrow{\text{incl}} TE \xrightarrow{\pi_*} \pi^*TX$ is an isomorphism, say η, and $s := \text{incl} \circ \eta^{-1}$ is a section.

Now take $E = T^*M$ and recall that $\omega_{\text{can}} = -d\lambda_{\text{can}}$ is a symplectic form on T^*M. With respect to ω_{can}, $T^{\text{vert}}T^*M$ is a Lagrangian subbundle, because λ_{can} is defined by first applying π_*, but $T^{\text{vert}}T^*M$ is by definition the kernel of π_*.

But we already know how to find a complement to a Lagrangian subbundle: Pick an ω_{can}-compatible almost complex structure J on TT^*M and set $V := J(T^{\text{vert}}T^*M)$. Then V is everywhere complementary to $T^{\text{vert}}T^*M$, hence gives us a splitting s. In fact, also V is Lagrangian with respect to ω_{can}, and of course $V \cong \pi^*TM$. Now

$$\Phi\colon\ \pi^*T^*M \oplus \pi^*TM \to TT^*M,\ (a,b) \mapsto \text{incl}(a) + s(b)$$

is the desired isomorphism. It satisfies everything we want except that we must check what $\Phi^*\omega_{\text{can}}$ is.

Thus fix a point $(x,\gamma) \in T^*M$, and let $(\alpha,v),(\beta,w) \in \pi^*(T^*M \oplus TM)_{(x,\gamma)}$, so $\alpha,\beta \in T_\gamma T^*_xM$ and $v,w \in T_xM$. Writing $\Phi(\alpha,v) = \Phi(\alpha,0) + \Phi(0,v)$ etc. and using the fact that both $T^{\text{vert}}T^*M$ and V are Lagrangian, we find

$$\phi^*\omega_{\text{can}}(\Phi(\alpha,v),\Phi(\beta,w)) = \omega_{\text{can}}(\Phi(\alpha,0),\Phi(0,w)) - \omega_{\text{can}}(\Phi(\beta,0),\Phi(0,v))$$

So it's enough to calculate $\omega_{\text{can}}(\Phi(\alpha,0),\Phi(0,w))$, which we can also write as $\omega_{\text{can}}(\alpha,s(w))$ by how we defined things.

Claim. $\omega_{\text{can}}(\alpha,s(w)) = -\alpha(w)$.

Here we identify $T_\gamma T^*_x M \cong T^*_x M$ by a canonical isomorphism and view α as an element of $T^*_x M$. Once we have proved the claim we have

$$\phi^*\omega_{\text{can}}(\Phi(\alpha, v), \Phi(\beta, w)) = \beta(v) - \alpha(w)$$

which is the canonical symplectic structure on $T^*_x M \oplus T_x M$.

In retrospect it would have been easier to prove the claim by just writing out ω_{can} in local coordinates as we did in the lecture...

Proof of the claim. Let $U \subset T^*M$ be a small neighbourhood of our chosen basepoint (x, γ). On that neighbourhood we choose local sections extending α and w, i.e. a vertical vector field

$$A\colon\ U \to T^{\text{vert}}T^*M|_U \subset TT^*M|_U$$

such that $A(x, \gamma) = \alpha$, and a section $W\colon\ U \to TM|_U$ such that $W(x, \gamma) = w$. In fact, we can choose W so that it is invariant in the vertical direction (i.e. independent of the argument γ). Using the formula for the exterior derivative we have

$$\begin{aligned}\omega_{\text{can}}(A, s(W)) &= -d\lambda_{\text{can}}(A, s(W)) \\ &= s(W)(\lambda_{\text{can}}(A)) - A(\lambda_{\text{can}}(s(W))) + \lambda_{\text{can}}([A, s(W)])\end{aligned}$$

Now λ_{can} is defined by first applying π_* and A is vertical, so the first term is certainly zero. But also the last term is zero. For example at the point $(x, \gamma) \in U$ we have

$$\pi_*([A, s(W)]_{(x,\gamma)}) = \pi_*\left(\frac{d}{dt}\Big|_{t=0}(\phi^A_{-t})_*(s(W)_{\phi^A_t(x,\gamma)})\right),$$

where ϕ^A_t is the local flow of A on U. But since A is vertical, this flow is vertical, so $\pi \circ \phi^A_t = \pi$. By the chain rule, because $\pi \circ s = id$, and because W is invariant in the vertical direction, we then get

$$\pi_*([A, s(W)]_{(x,\gamma)}) = \frac{d}{dt}\Big|_{t=0} W_{\phi^A_t(x,\gamma)} = 0\,.$$

The only term left is $-A(\lambda_{\text{can}}(s(W)))$. By definition of λ_{can}, $\lambda_{\text{can}}(s(W))$ is simply the function $U \to \mathbb{R}$, $(y, \tau) \mapsto \tau(W_{(y,\tau)})$. Therefore,

$$-A(\lambda_{\text{can}}(s(W)))_{(x,\gamma)} = -\frac{d}{dt}\Big|_{t=0}(\gamma + t\alpha)(W_{(x,\gamma)}) = -\alpha(w)\,.$$

$\square$