Thus, $\mathcal{L}(V, \omega, W) \cong$ $\mathbb{R}$-vector space of symmetric $n \times n$ matrices

$$\cong \mathbb{R}^{n+(n-1)+\dots+1} = \mathbb{R}^{\frac{1}{2}n(n+1)}.$$

Ex. 4. (1) If $\omega|_W \equiv 0$ then any basis will do.

Else, there are $u_1, v_1 \in W$ such that $\omega(u_1, v_1) = 1$ (normalize)

Necessarily u_1, v_1 are lin. indep.

Set $K = \operatorname{span}(u_1, v_1) \subseteq W$

Claim: $W = K \oplus K^\perp$.

Proof: if $t \in K \cap K^\perp$, write $t = \lambda u_1 + \mu v_1$

then $\underset{t \in K^\perp}{0} = \omega(t, v_1) = \lambda\, \omega(u_1, v_1) \Rightarrow \lambda = 0$

similarly, $\mu = 0$. $\square$

Now continue with $(K^\perp, \omega|_{K^\perp})$ and do induction $\square$

(2) • W is symplectic: Basis looks like $u_1, \dots, u_k, v_1, \dots, v_k$

Now $V = W \oplus W^\perp$ by non-deg. of $\omega|_W$

and $W^\perp$ is also symplectic by non-deg. of ω

The union of the given basis with a symplectic basis for $W^\perp$ is a symplectic basis for V.

- <u>W isotropic</u>: Basis looks like $w_1, \dots, w_p$.

  If $p = 1$ we're done.

  Else, there must be $t_1 \in \langle w_2, \dots, w_p \rangle^\perp$ with $\omega(w_1, t_1) = 1$

  for otherwise $w_1 \in (\langle w_2, \dots, w_p \rangle^\perp)^\perp = \langle w_2, \dots, w_p \rangle$ ↯

  Now $\{w_1, t_1\}$ can be extended to a symplectic basis

  $w_1, t_1, u_1, v_1, \dots, u_k, v_k$ for V.

  Moreover, $\langle w_2, \dots, w_p \rangle \subseteq \langle u_1, v_1, \dots, u_k, v_k \rangle = \langle w_1, t_1 \rangle^\perp$

  is an isotropic subspace of one dimension less.

  Now proceed by induction. □

- <u>W coisotropic</u> $W^\perp \subseteq W$, so $W^\perp$ is spanned by $w_1, \dots, w_p$.

  Now $\langle u_1, v_1, \dots, u_k, v_k \rangle$ is symplectic and its orthogonal

  complement which is symplectic contains the isotropic subspace $\langle w_1, \dots, w_p \rangle$.

  We already know how to extend this to a symplectic

  basis. □