Winter term 2023/24 16 January 2024

Topology I

Sheet 12

Exercise 1. Let $n \geq 1$ and let there be given a continuous map $f\colon S^n \to S^n$. Prove:

(a) If $f(x) \neq x$ for all $x \in S^n$, then $\deg(f) = (-1)^{n+1}$.

(b) If $f(x) \neq -x$ for all $x \in S^n$, then $\deg(f) = 1$.

(c) If n is even, then every map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$ has a fixed point, i.e, a point $z \in \mathbb{R}P^n$ with $g(z) = z$.

(d) If n is odd, then there exists a map $g\colon \mathbb{R}P^n \to \mathbb{R}P^n$ without fixed points.

Exercise 2. Recall that, given a space X and an integer $n \geq 0$, the abelian group of singular n-simplices in X is

$$C_n^{\mathrm{sing}}(X) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{Top}}(\Delta^n_{\mathrm{Top}}, X)].$$

Recall further the natural homomorphism $d_n\colon C_n^{\mathrm{sing}}(X) \to C_{n-1}^{\mathrm{sing}}(X)$.

(a) Show that the natural transformation $d_n\colon C_n^{\mathrm{sing}}(-) \to C_{n-1}^{\mathrm{sing}}(-)$ corresponds, via the Yoneda lemma, to an element $\partial\Delta^n_{\mathrm{Top}} \in C_{n-1}^{\mathrm{sing}}(\Delta^n_{\mathrm{Top}})$.

(b) Determine $\partial\Delta^n_{\mathrm{Top}}$.

Exercise 3. Let R be a ring (associative, with unit). Recall that a sequence of left R-modules

$$M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} M_n$$

is called *exact* if $\mathrm{im}(f_{i-1}) = \ker(f_i)$ for all $i = 1, \ldots, n-1$. Given left R-modules F and P, prove that the following are equivalent:

(a) There exists a map $f\colon F \to F$ with $f^2 = f$ and $\mathrm{im}(f) \cong P$.

(b) There is an exact sequence of left R-modules

$$0 \to K \to F \xrightarrow{g} P \to 0$$

and a map $s\colon P \to F$ such that $gs = id_P$.

(c) There is a left R-module K and an isomorphism $F \cong K \oplus P$.

(please turn)

Exercise 4. Let R be a commutative ring and $0 \to M \to N \to L \to 0$ an exact sequence of R-modules. Let K be another R-module. Prove:

(a) The induced sequence $M \otimes_R K \to N \otimes_R K \to L \otimes_R K \to 0$ is exact.

(b) The induced sequence $0 \to \mathrm{Hom}_R(K, M) \to \mathrm{Hom}_R(K, N) \to \mathrm{Hom}_R(K, L)$ is exact.

(c) The induced sequences $0 \to M \otimes_R K \to N \otimes_R K$ and $\mathrm{Hom}_R(K, N) \to \mathrm{Hom}_R(K, L) \to 0$ are not, in general, exact.

Recall that an R-module K is called *flat* if, for any exact sequence $0 \to M \to N \to L \to 0$, the induced sequence $0 \to M \otimes_R K \to N \otimes_R K$ is exact, and *projective* if for any such sequence $\mathrm{Hom}_R(K, N) \to \mathrm{Hom}_R(K, L) \to 0$ is exact. Also recall that K is *free* if it is of the form $K \cong \bigoplus_{s \in S} R$ for some set S.

(d) Show that, for an R-module P, the following are equivalent:

  (i) P is projective.

  (ii) Given a surjective map $f\colon N \to L$ and a map $g\colon P \to L$ there is a map $\bar{g}\colon P \to N$ such that $f\bar{g} = g$.

  (iii) There exist R-modules F and K, with F free, and an isomorphism $F \cong K \oplus P$.

(e) Show that a free R-module is projective, and a projective R-module is flat.

(f*) Let I be a small filtered category and $\mathrm{Fun}(I, \mathrm{Mod}(R))$ the category of I-shaped diagrams of R-modules. Show that $\mathrm{colim}\colon \mathrm{Fun}(I, \mathrm{Mod}(R)) \to \mathrm{Mod}(R)$ is an exact functor, i.e., given an exact sequence $0 \to F \to G \to H \to 0$ in $\mathrm{Fun}(I, \mathrm{Mod}(R))$, the induced sequence of R-modules $0 \to \mathrm{colim}(F) \to \mathrm{colim}(G) \to \mathrm{colim}(H) \to 0$ is exact. Conclude that if $F \in \mathrm{Fun}(I, \mathrm{Mod}(R))$ with $F(i)$ flat for every $i \in I$, then $\mathrm{colim}(F)$ is flat.

This sheet will be discussed in the week of 22 January 2024.