LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

Winter term 2023/24 — 24 October 2023

Topology I

Sheet 2

Exercise 1. Let X be a topological space and $A \subseteq X$ a subspace.

(a) Under what conditions on A is the canonical map $X \to X/A$ open?

(b) Let $B \subseteq A$ be another subspace. Show that A/B is naturally a subspace of X/B and there is a homeomorphism $(X/B)/(A/B) \cong X/A$.

Exercise 2. Let G be a group acting continuously on a topological space X.

(a) Show that the canonical map $X \to X/G$ is open.

(b) Suppose that $H \leq G$ is a normal subgroup. Show that G/H acts continuously on X/H and there is a homeomorphism $(X/H)/(G/H) \cong X/G$.

Exercise 3. Consider the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}$ for $\mathbb{Z}$ acting on itself via sign: The underlying set of $\mathbb{Z} \rtimes \mathbb{Z}$ is $\mathbb{Z} \times \mathbb{Z}$ and the group law $\cdot$ is defined by

$$(a, b) \cdot (a', b') = (a + (-1)^b a', b + b') \quad (a, b), (a', b') \in \mathbb{Z}^2 .$$

(a) Show that $\mathbb{Z} \rtimes \mathbb{Z}$ is generated by $S = (1, 0)$ and $T = (0, 1)$, and that $\mathbb{Z} \rtimes \mathbb{Z}$ acts continuously on $\mathbb{R}^2$ by $S(x, y) = (x, y + 1)$ and $T(x, y) = (x + 1, -y)$ for all $(x, y) \in \mathbb{R}^2$. The quotient $K = \mathbb{R}^2/\mathbb{Z} \rtimes \mathbb{Z}$ is called the Klein bottle.

(b) Exhibit $\mathbb{Z}^2$ as an index two subgroup of $\mathbb{Z} \rtimes \mathbb{Z}$.

(c) Let C_2 be a cyclic group of order two. Show that there is a continuous action of C_2 on the 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and a map $T^2 \to K$ inducing a homeomorphism $T^2/C_2 \cong K$.

(d) Show that the unit square $[0, 1]^2 \subseteq \mathbb{R}^2$ is a fundamental domain for the action of both $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes \mathbb{Z}$. Describe both T^2 and K as quotient spaces of $[0, 1]^2$ by a suitable equivalence relation.

Exercise 4. Let X be a topological space and $f\colon X \to X$ a map. Define the *mapping torus of* f to be the space $T_f = (X \times [0, 1])/\sim$ where $\sim$ is the equivalence relation generated by $(x, 1) \sim (f(x), 0)$ for all $x \in X$.

(a) Show that the projection onto $[0, 1]$ induces a continuous map $T_f \to S^1$.

(b) Show that the Klein bottle K is homeomorphic to the mapping torus of $f\colon S^1 \to S^1$, $f(z) = z^{-1}$.

(c) Describe the composite map $T^2 \to K \to S^1$ (cf. Exercise 3 (c)).

(please turn)

Exercise 5. Let G be a group acting continuously on a space X. Let $f_1, f_2\colon\ G \times X \to X$ be the maps defined by $f_1(g,x) = x$ and $f_2(g,x) = gx$, respectively. Compute the coequaliser of f_1 and f_2.

Exercise 6. Let $p > 1$ be an integer and let $q_1, \ldots, q_n$ be integers coprime to p. An action of the cyclic group C_p on $\mathbb{C}^n$ is generated by

$$(z_1, \ldots, z_n) \mapsto (e^{2\pi i q_1/p} z_1, \ldots, e^{2\pi i q_n/p} z_n)\,.$$

In the lecture we wrote $\mathbb{C}(q_1) \oplus \cdots \oplus \mathbb{C}(q_n)$ for $\mathbb{C}^n$ equipped with this action. Show that the induced action on the unit sphere $S(\mathbb{C}(q_1) \oplus \cdots \oplus \mathbb{C}(q_n)) \cong S^{2n-1}$ is free.

This sheet will be discussed in the week of 30 October 2023.