LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
MÜNCHEN

MATHEMATISCHES INSTITUT

Winter term 2023/24 17 October 2023

# Topology I

Sheet 1

**Exercise 1.** Let $\mathcal{O}$ be the cofinite topology on the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Prove that $(\mathbb{N}, \mathcal{O})$ is not metrizable by showing the following:

(a) Any two non-empty open sets $U, V \in \mathcal{O}$ have non-empty intersection.

(b) If $X$ is a non-empty metrizable space for which any two non-empty open sets have non-empty intersection, then $|X| = 1$, i.e., $X$ is a one-point space.

**Exercise 2.** Let $X$ be a set and let $T \subseteq \mathcal{P}(X)$ be a set of subsets of $X$. Prove that

$$\mathcal{S}_T = \left\{ \bigcup_{i \in I} A_i \mid I \text{ is a set and } \forall i \in I \text{ there is a finite subset } J_i \subseteq T \text{ such that } A_i = \bigcap_{B \in J_i} B \right\}$$

is a topology on $X$.

**Exercise 3.** Let $\mathcal{O}$ and $\mathcal{O}'$ be two topologies on a set $X$. Decide under which condition on $\mathcal{O}$ and $\mathcal{O}'$ the identity map $id\colon X \to X$ is continuous with respect to $\mathcal{O}$ and $\mathcal{O}'$.

**Exercise 4.** Let $X = \{a, b\}$ be a set with two elements.

(a) Give a list of all topologies on $X$, and decide which ones are homeomorphic.

(b) Consider the unit interval $[0, 1] \subseteq \mathbb{R}$ with the standard topology. Describe the quotient topology on $[0, 1]/[0, 1)$, and decide which of the topologies in (a) it corresponds to.

**Exercise 5.** Let $X$ be a topological space and $I$ a set. Let there be given for each $i \in I$ a subset $A_i \subseteq X$ such that $X = \bigcup_{i \in I} A_i$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$. Prove that the canonical map $\amalg_{i \in I} A_i \to X$ is a homeomorphism if and only if each $A_i$ is open and closed in $X$.

This sheet will be discussed in the week of 23 October 2023.