COMMENTS ON SHEET 8

Exercise 1

First note that there is a deformation retraction of $\mathbb{R}^2$ onto $[-1,1]^2$ defined as follows:

$$\begin{aligned} h_o\colon\ \mathbb{R}^2 \times [0,1] &\to \mathbb{R}^2 \\ (x,t) &\mapsto \begin{cases} \frac{x}{(1-t)+t||x||_\infty} & \text{if } ||x||_\infty > 1 \\ x & \text{if } ||x||_\infty \le 1 \end{cases} \end{aligned}$$

where $||\cdot||_\infty$ is the maximum norm on $\mathbb{R}^2$. Similarly, there is a deformation retraction of $[-1,1]^2\backslash\{(0,0)\}$ onto $\partial[-1,1]^2$ defined by

$$\begin{aligned} h_i\colon\ [-1,1]^2\backslash\{(0,0)\} \times [0,1] &\to [-1,1]^2\backslash\{(0,0)\} \\ (x,t) &\mapsto \frac{x}{(1-t)+t||x||_\infty} \end{aligned}$$

Let $A_k = [k,k+1]\times[0,1]$, $k=0,\dots,n-1$ and $A = \bigcup_{k=0}^{n-1} A_k \subseteq \mathbb{R}^2$. Note that $S \subseteq A$. Let $\phi\colon (\mathbb{R}^2, A) \to (\mathbb{R}^2, [-1,1]^2)$ be a relative homeomorphism, i.e., a homeomorphism of $\mathbb{R}^2$ that restricts to a homeomorphism from A onto $[-1,1]^2$. Then

$$\mathbb{R}^2 \times [0,1] \xrightarrow{\phi\times id} \mathbb{R}^2 \times [0,1] \xrightarrow{h_o} \mathbb{R}^2 \xrightarrow{\phi^{-1}} \mathbb{R}^2$$

is a deformation retraction of $\mathbb{R}^2$ onto A, giving a homotopy equivalence $\mathbb{R}^2\backslash S \simeq A\backslash S$.

Let $p_k = (\frac{2k+1}{2}, \frac{1}{2}) \in A_k$ and let $\psi_k\colon (A_k, p_k) \to ([-1,1]^2, (0,0))$ be a relative homeomorphism. Then

$$A_k\backslash\{p_k\} \times [0,1] \xrightarrow{\psi_k\times id} [-1,1]^2\backslash\{(0,0)\} \times [0,1] \xrightarrow{h_i} [-1,1]^2\backslash\{(0,0)\} \xrightarrow{\psi_k^{-1}} A_k\backslash\{p_k\}$$

is a deformation retraction of $A_k\backslash\{p_k\}$ onto its boundary. The ψ_k, $k=0,\dots,n-1$ combine to give a deformation retraction of $A\backslash S$ onto the graph $G = [0,n]\times\{0,1\} \cup \bigsqcup_{k=0}^{n}\{k\}\times[0,1]$. But G is homotopy equivalent to a wedge of n circles by Exercise 3, Sheet 6.

Exercise 2

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By radial projection onto the circumference of the polygon we may assume that Σ_g is obtained as a quotient of a disc D^2, whose boundary circle is divided into $4g$ arcs of equal length, by an equivalence relation $\sim$ that identifies the boundary arcs in the obvious way. In particular, there is a homeomorphism $\phi\colon \partial D^2/\sim \cong \bigvee^{2g} S^1$.

The attaching map $\alpha\colon S^1 \to \bigvee^{2g} S^1$ is then simply the composite of $S^1 = \partial D^2 \to \partial D^2/\sim$ with ϕ. The projection $D^2 \to D^2/\sim$ and the map $\phi^{-1}\alpha\colon S^1 \to \partial D^2/\sim$ induce a continuous bijection

$$\mathrm{pushout}(D^2 \xleftarrow{i} S^1 \xrightarrow{\alpha} \bigvee^{2g} S^1) \to D^2/\sim = \Sigma_g$$

It is a homeomorphism, because the domain is compact and the codomain is Hausdorff.

Exercise 3

(a) For every $x \in X$ and $g \in G$, $g \neq e$ there is an open $x \in U \subseteq X$ such that $U \cap gU = \emptyset$. Indeed, the action is free, so $x \neq gx$, and X is Hausdorff, so there are open neighbourhoods V_1 of x and V_2 of gx such that $V_1 \cap V_2 = \emptyset$. Then define $U := g^{-1}(g(V_1) \cap V_2)$.

Now let $x \in X$ be fixed. Since X is locally compact, we can pick a compact neighbourhood $x \in K \subseteq X$. Since the action is proper, the set

$$S = \{g \in G\backslash\{e\} \mid g(K) \cap K \neq \emptyset\}$$

is finite. By the previous paragraph, we find for each $g \in S$ an open neighbourhood $x \in U_g \subseteq K$ such that $g(U_g) \cap U_g = \emptyset$. Then $U := \bigcap_{g\in S} U_g$ is an open neighbourhood of x such that $g(U) \cap U = \emptyset$ for all $g \in G\backslash\{e\}$. This means that the action is covering-like.

(b) Let X be non-empty and $X \sqcup X$ as in the hint. The action of C_2 on $X \sqcup X$ is obviously continuous, free, and it is proper because C_2 is finite. But it is not covering like, because for any open $U \sqcup U \subseteq X \sqcup X$ we have that $\tau(U \sqcup U) = U \sqcup U$, where $\tau \in C_2$ is the non-trivial element.

Concretely, we can choose $X = *$, then $X \sqcup X$ is the space with two points and indiscrete topology. This satisfies all the assumptions of (a) except it is not Hausdorff. The fold map $* \sqcup * \to *$ (which is the projection wrt to the C_2-action) is not a covering map, and hence C_2 does not act covering-like: if it were a covering map, it would be a trivial covering (because the base space is a single point), i.e., $* \sqcup * \cong * \amalg *$ (the coproduct), but this is not the case.

Exercise 4

(a) To show that $p\colon E \to B$ is open we show that every $x \in E$ has an open neighbourhood $V \subseteq E$ which is mapped by p homeomorphically onto an open subset $p(V) \subseteq B$. By definition of a covering, $p(x)$ has an open neighbourhood

$U \subseteq B$ such that $p|_{p^{-1}(U)}$ is a trivial covering, i.e., there is a homeomorphism $\varphi\colon\ p^{-1}(U) \cong U \times F$ over U with F discrete. Let $z \in F$ such that $\varphi(x) \in U \times \{z\}$. Then $V := \varphi^{-1}(U \times \{z\})$ is an open neighbourhood of x and it is mapped by p homeomorphically onto U.

(b) Let $U \subseteq B$ and suppose that $p^{-1}(U) \subseteq E$ is open. As p is surjective, $U = p(p^{-1}(U))$, and this is open, because p is open by (a). It follows that p is a quotient map. (This shows that an open surjective map is a quotient map.)

(c) We will show that $\mathrm{im}(p)$ is open and closed and non-empty.

Openness was shown in (a).

The image is non-empty, because E is assumed non-empty.

To show that $\mathrm{im}(p)$ is closed, let $x \in B \backslash \mathrm{im}(p)$. By definition of a covering, there is an open subset $x \in U \subseteq B$ and a homeomorphism $p^{-1}(U) \cong U \times F$ over U with F discrete. But since $x \notin \mathrm{im}(p)$, we must have $F = \emptyset$. Hence, $p^{-1}(U) = \emptyset$, so $U \subseteq B \backslash \mathrm{im}(p)$ showing that $B \backslash \mathrm{im}(p)$ is open.

(d) Let $x \in E$ and let $V \subseteq E$ be an open neighbourhood of x. By (a), $p(V)$ is an open neighbourhood of $p(x)$. Since B is locally (path-)connected, we find a (path-)connected neighbourhood $U \subseteq p(V)$ of $p(x)$. Wlog, we may assume U is such that $p|_{p^{-1}(U)}$ is a trivial covering, i.e., there is a homeomorphism $\varphi\colon\ p^{-1}(U) \cong U \times F$ over U with F discrete. If $z \in F$ is such that $\varphi(x) \in U \times \{z\}$, then $\varphi^{-1}(U \times \{z\}) \subseteq V$ is a (path-)connected neighbourhood of x.

With “globally” insetad of “locally” this is not true: Just take a trivial covering with more than one sheet and (path-)connected base space.

Exercise 5

(a) We'll show that if $p\colon\ E \to B$ is a trivial covering, i.e., if there is a discrete space F and a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow[\cong]{\varphi} & B \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & B & \end{array}$$

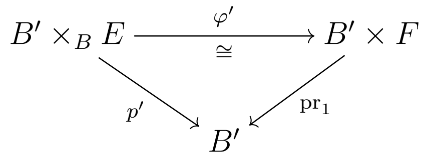
then $p'\colon\ B' \times_B E \to B'$ is a trivial covering, too. Indeed, the map

$$\begin{aligned} \varphi'\colon\ B' \times_B E &\to B' \times F \\ (x, u) &\mapsto (x, \mathrm{pr}_2(\varphi(u))) \end{aligned}$$

is a homeomorphism with inverse

$$\begin{aligned} B' \times F &\to B' \times_B E \\ (x, v) &\mapsto (x, \varphi^{-1}(f(x), v)) \end{aligned}$$

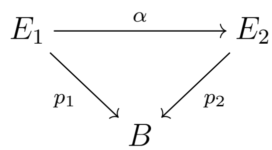
and clearly, the diagram



commutes.

Now suppose that $p\colon E \to B$ is any covering map. Given $x \in B'$ there is an open neighbourhood $U \subseteq B$ of $f(x)$ such that p is trivial over U. But then, by the previous paragraph, p' is trivial over $f^{-1}(U)$. So p' is a covering map.

(b) Let $\alpha\colon (p_1, E_1, B) \to (p_2, E_2, B)$ be a map of coverings, i.e., $\alpha\colon E_1 \to E_2$ is a map such that



commutes. Define

$$\begin{aligned} f^*(\alpha)\colon\ B' \times_B E_1 &\to B' \times_B E_2 \\ (x, u) &\mapsto (x, \alpha(u)) \end{aligned}$$

This is well-defined, because $f(x) = p_1(x) = p_2(\alpha(u))$, as required.

Clearly, $f^*(\alpha)$ is continuous and satisfies $p_2' \circ f^*(\alpha) = p_1'$, so $f^*(\alpha)$ is a map of coverings. It is also clear that $f^*(id) = id$ and $f^*(\alpha\beta) = f^*(\alpha)f^*(\beta)$.