COMMENTS ON SHEET 7

Exercise 1

The induced functor $L\colon \mathcal{C}_1\hat{\times}_{\mathcal{C}_0}\mathcal{C}_2 \to \mathcal{D}_1\hat{\times}_{\mathcal{D}_0}\mathcal{D}_2$ is defined by

$$L(X_1, X_2, \alpha\colon F_1(X_1) \cong F_2(X_2)) = (L_1(X_1), L_2(X_2), L_0(\alpha)\colon L_0(F_1(X_1)) \cong L_0(F_2(X_2)))$$

on objects (note that $L_0 \circ F_1 = G_1 \circ L_1$ and $L_0 \circ F_2 = G_2 \circ L_2$ so the latter is indeed an object of $\mathcal{D}_1\hat{\times}_{\mathcal{D}_0}\mathcal{D}_2$) and by

$$L(f_1, f_2) = (L_1(f_1), L_2(f_2))$$

on morphisms.

To see that L is faithful if L_1 and L_2 are faithful note the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}((X_1, X_2, \alpha), (X_1', X_2', \alpha')) & \xrightarrow{L} & \mathrm{Hom}((L_1(X_1), L_2(X_2), L_0(\alpha)), (L_1(X_1'), L_2(X_2'), L_0(\alpha'))) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}(X_1, X_1') \times \mathrm{Hom}(X_2, X_2') & \xrightarrow{L_1 \times L_2} & \mathrm{Hom}(L_1(X_1), L_1(X_1')) \times \mathrm{Hom}(L_2(X_2), L_2(X_2')) \end{array}$$

where the two vertical arrows are inclusions of subsets. Thus, if L_1 and L_2 are injective, then so is L.

Now assume that L_1 and L_2 are full and L_0 is faithful. Let

$$(g_1, g_2) \in \mathrm{Hom}((L_1(X_1), L_2(X_2), L_0(\alpha)), (L_1(X_1'), L_2(X_2'), L_0(\alpha'))) .$$

Since L_1 and L_2 are full, there is $(f_1, f_2) \in \mathrm{Hom}(X_1, X_1') \times \mathrm{Hom}(X_2, X_2')$ such that $L_1(f_1) = g_1$ and $L_2(f_2) = g_2$. To show that (f_1, f_2) lies in the subset $\mathrm{Hom}((X_1, X_2, \alpha), (X_1', X_2', \alpha'))$ we must show that the diagram

$$\begin{array}{ccc} F_1(X_1) & \xrightarrow{F_1(f_1)} & F_1(X_1') \\ \cong \big\downarrow \alpha & & \cong \big\downarrow \alpha' \\ F_2(X_2) & \xrightarrow{F_2(f_2)} & F_2(X_2') \end{array}$$

commutes. Since L_0 is faithful, we can check commutativity after applying L_0 to the diagram. But after applying L_0 the diagram does commute, because (g_1, g_2) is a morphism from $(L_1(X_1), L_2(X_2), L_0(\alpha))$ to $(L_1(X_1'), L_2(X_2'), L_0(\alpha'))$. Thus, L is full.

Now assume that L_1 and L_2 are essentially surjective and L_0 is fully faithful. Let $(Y_1, Y_2, \alpha\colon G_1(Y_1) \cong G_2(Y_2))$ be an object of $\mathcal{D}_1\hat{\times}_{\mathcal{D}_0}\mathcal{D}_2$. Since L_1 and L_2

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are essentially surjective, we find $X_i \in \mathcal{C}_i$ and isomorphisms $\beta_i \colon Y_i \cong L_i X_i$ for $i = 1, 2$. Now we have the solid portion of the following diagram

$$\begin{array}{ccc} G_1(Y_1) & \xrightarrow[\cong]{G_1(\beta_1)} & G_1(L_1(X_1)) \\ \cong \big\downarrow \alpha & & \big\downarrow L_0(\bar{\alpha}) \\ G_2(Y_2) & \xrightarrow[\cong]{G_2(\beta_2)} & G_2(L_2(X_2)) \end{array}$$

and we must find an isomorphism $\bar{\alpha} \colon F_1(X_1) \cong F_2(X_2)$ such that

$$L_0(\bar{\alpha}) \colon L_0(F_1(X_1)) = G_1(L_1(X_1)) \xrightarrow{\cong} L_0(F_2(X_2)) = G_2(L_2(X_2))$$

makes the diagram commute. Indeed, then (β_1, β_2) would be an isomorphism from (Y_1, Y_2, α) to $(L_1(X_1), L_2(X_2), L_0(\bar{\alpha}))$ and the latter object is in the image of L.

Since L_0 is full, there exists $\bar{\alpha} \colon F_1(X_1) \to F_2(X_2)$ such that

$$L_0(\bar{\alpha}) = G_2(\beta_2) \circ \alpha \circ G_1(\beta_1)^{-1},$$

and this choice obviously makes the diagram commute. Since L_0 is fully faithful, $\bar{\alpha}$ is an isomorphism, too.

It follows that L is essentially surjective.

Exercise 2

An object of $\mathrm{Fun}(B\mathbb{Z}, \mathcal{G})$ is a pair (x, γ), where x is an object of $\mathcal{G}$ and γ is an automorphism of x. A morphism $f \colon (x_1, \gamma_1) \to (x_2, \gamma_2)$ is an isomorphism $f \colon x_1 \cong x_2$ such that the diagram

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ \big\downarrow \gamma & & \big\downarrow \gamma_2 \\ x_1 & \xrightarrow{f} & x_2 \end{array}$$

commutes. Define a functor $I \colon \mathrm{Fun}(B\mathbb{Z}, \mathcal{G}) \to \mathcal{G} \hat{\times}_{\mathcal{G}\times\mathcal{G}} \mathcal{G}$ by

$$I(x, \gamma) = (x, x, (\gamma, id))$$

on objects and by $I(f) = (f, f)$ on morphisms. To see that I is an equivalence we define an inverse equivalence $P \colon \mathcal{G} \hat{\times}_{\mathcal{G}\times\mathcal{G}} \mathcal{G} \to \mathrm{Fun}(B\mathbb{Z}, \mathcal{G})$ by

$$P(x, y, (\alpha, \beta)) = (x, \beta^{-1}\alpha)$$

on objects and by $I(f, g) = f$ on morphisms. Let us check that this is indeed well-defined: Suppose $(f, g) \colon (x_1, y_1, (\alpha_1, \beta_1)) \to (x_2, y_2, (\alpha_2, \beta_2))$ is a morphism in $\mathcal{G}\hat{\times}_{\mathcal{G}\times\mathcal{G}}\mathcal{G}$. This means that the following diagram commutes

$$\begin{array}{ccc} (x_1, x_1) & \xrightarrow{(f,f)} & (x_2, x_2) \\ \downarrow{\scriptstyle (\alpha_1,\beta_1)} & & \downarrow{\scriptstyle (\alpha_2,\beta_2)} \\ (y_1, y_1) & \xrightarrow{(g,g)} & (y_2, y_2) \end{array}$$

To see that $P(f, g) = f\colon (x_1, \beta_1^{-1}\alpha_1) \to (x_2, \beta_2^{-1}\alpha_2)$ is indeed a morphism in $\mathrm{Fun}(B\mathbb{Z}, \mathcal{G})$ we must check that the diagram

$$\begin{array}{ccc} x_1 & \xrightarrow{f} & x_2 \\ \downarrow{\scriptstyle \beta_1^{-1}\alpha_1} & & \downarrow{\scriptstyle \beta_2^{-1}\alpha_2} \\ x_1 & \xrightarrow{f} & x_2 \end{array}$$

commutes. And indeed,

$$\beta_2^{-1}\alpha_2 f = \beta_2^{-1} g\alpha_1 = f\beta_1^{-1}\alpha_1$$

by commutativity of the previous diagram.

It is clear that $PI = id_{\mathrm{Fun}(B\mathbb{Z},\mathcal{G})}$. The other composite IP sends a morphism $(f, g)\colon (x_1, y_1, (\alpha_1, \beta_1)) \to (x_2, y_2, (\alpha_2, \beta_2))$ to the morphism

$$(f, f)\colon (x_1, x_1, (\beta_1^{-1}\alpha_1, id)) \to (x_2, x_2, (\beta_2^{-1}\alpha_2, id))\,.$$

We must find a natural isomorphism $id_{\mathcal{G}\hat{\times}_{\mathcal{G}\times\mathcal{G}}\mathcal{G}} \cong IP$. The following diagrams constitute such a natural isomorphism:

$$\begin{array}{ccc} (x_1, y_1, (\alpha_1, \beta_1)) & \xrightarrow{(id,\beta_1^{-1})} & (x_1, x_1, (\beta_1^{-1}\alpha_1, id)) \\ \downarrow{\scriptstyle (f,g)} & & \downarrow{\scriptstyle (f,f)} \\ (x_2, y_2, (\alpha_2, \beta_2)) & \xrightarrow{(id,\beta_2^{-1})} & (x_2, x_2, (\beta_2^{-1}\alpha_2, id)) \end{array}$$

(i.e., (id, β_1^{-1}) is the component of the natural isomorphism at $(x_1, y_1, (\alpha_1, \beta_1))$).

Exercise 3

Let $G\colon \mathcal{G}_1 \to \mathcal{G}$ and $H\colon \mathcal{G}_0 \to \mathcal{G}$ be functors and $\eta\colon GF \cong H$ a natural isomorphism. Define a functor $\tilde{H}\colon \mathcal{G}_1 \to \mathcal{G}$ and a natural isomorphism $\tilde{\eta}\colon G \cong \tilde{H}$ as follows:

- For objects $F(x) \in \mathcal{G}_1$ we set $\tilde{H}(F(x)) := H(x)$ and $\tilde{\eta}_{F(x)} := \eta_x$.
- For objects $y \in \mathcal{G}_1$ that are not in the image of F we set $\tilde{H}(y) := G(y)$ and $\tilde{\eta}_y := id$.

- For any morphism $f\colon\ y \to z$ in $\mathcal{G}_1$ we define $\tilde{H}(f)\colon\ \tilde{H}(y) \to \tilde{H}(z)$ so that the following diagram commutes:

$$\begin{array}{ccc} G(y) & \xrightarrow{\tilde{\eta}_y} & \tilde{H}(y) \\ \Big\downarrow{\scriptstyle G(f)} & & \Big\downarrow{\scriptstyle \tilde{H}(f)} \\ G(z) & \xrightarrow{\tilde{\eta}_z} & \tilde{H}(z) \end{array}$$

This $\tilde{H}$ does the job: It is a functor, and by construction $\tilde{\eta}\colon\ G \cong \tilde{H}$ is a natural isomorphism.

If the morphism $f\colon\ y \to z$ is in the image of F, i.e., $f = F(f')\colon\ F(y') \to F(z')$, then the above diagram reads

$$\begin{array}{ccc} G(F(y')) & \xrightarrow{\eta_{y'}} & H(y') \\ \Big\downarrow{\scriptstyle G(F(f'))} & & \Big\downarrow{\scriptstyle \tilde{H}(F(f'))} \\ G(F(z')) & \xrightarrow{\eta_{z'}} & H(z') \end{array}$$

and because $\eta\colon\ GF \cong H$ is a natural isomorphism, this implies that $\tilde{H}(F(f')) = H(f')$. So $F^*(\tilde{H}) = H$.

Finally, $F^*(\tilde{\eta}) = \eta$ (the natural transformation $F^*(\tilde{\eta})$ is by definition the one with $F^*(\tilde{\eta})_x := \tilde{\eta}_{F(x)}$).

Exercise 4

Let $A \subseteq X$ be half of a great circle connecting the north pole and the south pole. Let $U \subseteq X$ be a small open neighbourhood of the subspace $A \cup C \subseteq X$ and let $V \subseteq X$ be a small open neighbourhood of $S^2 \subseteq X$ (so V is S^2 together with two small segments of C sticking out of the north and the south pole, respectively).

Clearly, $U \cup V = X$, $U \simeq S^1$, $V \simeq S^2$ and $U \cap V \simeq *$. In particular, U, V and $U \cap V$ are path-connected, so we can apply the Seifert-van-Kampen theorem. Take $(0,0,1) \in U \cap V$ as the basepoint. Then, using the fact that $\pi_1(S^2, (0,0,1)) \cong 1$ and $\pi_1(S^1, 1) \cong \mathbb{Z}$, up to natural isomorphism, the pushout diagram of the Seifert-van-Kampen theorem looks like

$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{Z} \\ \Big\downarrow & & \Big\downarrow \\ 1 & \longrightarrow & \pi_1(X, (0,0,1)) \end{array}$$

It follows that $\pi_1(X, (0,0,1)) \cong \mathbb{Z}$.

(Of course, one could also observe that there is a homotopy equivalence $X \simeq S^2 \vee S^1$, and so it follows from Example 2.68 in the lecture notes that $\pi_1(X) \cong \mathbb{Z}$.)