COMMENTS ON SHEET 6

Exercise 1

Define a map $f\colon \mathbb{N} \to X$ by

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} & \text{if } n > 0 \end{cases}.$$

Because $\mathbb{N}$ is discrete, f is continuous.

We claim that f is a weak equivalence: Since X is a subspace of $\mathbb{Q}$ and $\mathbb{Q}$ is totally disconnected, so is X (cf. Exercise 2c, Sheet 3). So the path-connected components of X are the singletons. Because the same is true for $\mathbb{N}$, the induced map of sets

$$\pi_0(f)\colon \pi_0(\mathbb{N}) \to \pi_0(X)$$

is a bijection.

Next suppose that $k \geq 1$ and let $n \in \mathbb{N}$. An element of $\pi_k(\mathbb{N}, n)$ is represented by a based continuous map $\alpha\colon (S^k, 1) \to (\mathbb{N}, n)$. Since S^k is path-connected, so is its image $\alpha(S^k) \subseteq \mathbb{N}$. Since the path-connected component of n is $\{n\}$, α must be the constant map with value n. Therefore, $\pi_k(\mathbb{N}, n) = 0$. In the same way we see that $\pi_k(X, x) = 0$ for any $x \in X$. It follows that f induces an isomorphism of groups

$$\pi_k(f)\colon \pi_k(\mathbb{N}, n) \to \pi_k(X, f(n))$$

for all $k \geq 1$ and any choice of basepoint $n \in \mathbb{N}$. Thus, f is a weak equivalence.

Finally, we show that X and $\mathbb{N}$ are not homotopy equivalent. Assume, for contradiction, that $g\colon \mathbb{N} \to X$ is a homotopy equivalence with homotopy inverse $h\colon X \to \mathbb{N}$. Then, $hg \simeq id_{\mathbb{N}}$, but since $\mathbb{N}$ is discrete, we must actually have $hg = id_{\mathbb{N}}$. Since X is compact, $h(X) \subseteq \mathbb{N}$ is bounded, and therefore $hg \neq id_{\mathbb{N}}$, a contradiction.

Exercise 2

Let $j\colon D^n \hookrightarrow S^n$ be an embedding that identifies D^n with the closed upper hemisphere of S^n. Let $\bar{j}$ be the composite of j with the projection $S^n \to \mathbb{R}P^n$.

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Let $k\colon S^{n-1} \hookrightarrow S^n$ be the inclusion of the equator and let $\bar{k}\colon \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ be the map induced by k on quotients. Then we have a commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\pi} & \mathbb{R}P^{n-1} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{k}} \\ D^n & \xrightarrow{\bar{j}} & \mathbb{R}P^n \end{array}$$

Recall that the pushout of $D^n \xleftarrow{i} S^{n-1} \xrightarrow{\pi} \mathbb{R}P^{n-1}$ is constructed as the quotient of $D^n \amalg \mathbb{R}P^{n-1}$ by the equivalence relation generated by $i(x) \simeq \pi(x)$ for all $x \in S^{n-1}$. We obtain a continuous map

$$\varphi\colon\ (D^n \amalg \mathbb{R}P^{n-1})/\sim \to \mathbb{R}P^n$$

which restricts to $\bar{j}$ and to $\bar{k}$ along the obvious maps $D^n \to (D^n \amalg \mathbb{R}P^{n-1})/\sim$ and $\mathbb{R}P^{n-1} \to (D^n \amalg \mathbb{R}P^{n-1})/\sim$, respectively. It is easily checked that φ is a bijection. Since $(D^n \amalg \mathbb{R}P^{n-1})/\sim$ is compact and $\mathbb{R}P^n$ is Hausdorff, it is a homeomorphism. This proves the pushout in (a). It also shows how $\mathbb{R}P^{n-1}$ can be viewed as a (equatorial) subspace of $\mathbb{R}P^n$.

One defines a filtration on $\mathbb{R}P^n$ by $\mathrm{sk}_k(\mathbb{R}P^n) := \mathbb{R}P^k$. The pushouts show that $\mathrm{sk}_k(\mathbb{R}P^n)$ is obtained form $\mathrm{sk}_{k-1}(\mathbb{R}P^n)$ by attaching a single cell of dimension k. Thus, the filtration is a CW-structure on $\mathbb{R}P^n$ with exactly one cell in every dimension $0 \le k \le n$.

(b) The pushouts for $\mathbb{C}P^n$ and $\mathbb{H}P^n$ look like

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\pi} & \mathbb{C}P^{n-1} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{k}} \\ D^{2n} & \xrightarrow{\bar{j}} & \mathbb{C}P^n \end{array} \quad \text{and} \quad \begin{array}{ccc} S^{4n-1} & \xrightarrow{\pi} & \mathbb{H}P^{n-1} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \bar{k}} \\ D^{4n} & \xrightarrow{\bar{j}} & \mathbb{H}P^n \end{array}$$

They are proved in the same way as for $\mathbb{R}P^n$. Setting $\mathrm{sk}_k(\mathbb{C}P^n) := \mathbb{C}P^k$ we obtain a CW-structure on $\mathbb{C}P^n$ with exactly one $2k$-dimensional cell for every $0 \le k \le n$.

And setting $\mathrm{sk}_k(\mathbb{H}P^n) := \mathbb{H}P^k$, we obtain a CW-structure on $\mathbb{H}P^n$ with exactly one $4k$-cell for every $0 \le k \le n$.

Exercise 3

Exercise 4