COMMENTS ON SHEET 2

Exercise 1

(a) The quotient map $p\colon X \to X/A$ is not open in general. Let $U \subseteq X$ be an open set. If U is disjoint from A, then $p(U)$ does not contain the special point to which all of A has been identified, and so $p^{-1}(p(U)) = U$. By definition of the quotient topology this means that $p(U)$ is open. If U and A are not disjoint, then $p(U)$ does contain $[a], a \in A$, and so $p^{-1}(p(U)) = U \cup A$. The set $U \cup A$ is not open in general, even if U is (find an example). So $p(U)$ need not be open. It will be open, if A is open; so, if A is open, then p is open.

(b) First recall that a having a continuous injective map $i\colon A \to X$ does not mean that we can view A as a subspace of X via i - the subspace topology on $i(A) \subseteq X$ and the topology on A need not agree, i.e., the map $i\colon A \to X$ need not be a homeomorphism onto its image. If $i\colon A \to X$ is a homeomorphism onto its image, then we call i an embedding. If $A \subseteq X$ is a subspace, then the inclusion $i\colon A \to X$ is indeed an embedding, by definition of the subspace topology.

Now let $B \subseteq A \subseteq X$ be subspaces, let $i\colon A \to X$ be the inclusion, and let $p_A\colon A \to A/B$ and $p_X\colon X \to X/B$ be the canonical quotient maps.

There is an obvious continuous injective map $\bar{i}\colon A/B \to X/B$ induced by i upon passage to quotients. We claim that $\bar{i}$ is a homeomorphism onto its image; this will show that A/B can be viewed as a subspace of X/B via $\bar{i}$.

Clearly, $\bar{i}$ is a continuous bijection onto its image, so it suffices to show that $\bar{i}\colon A/B \to \bar{i}(A/B)$ is open. To this end, let $U \subseteq A/B$ be open. Then $p_A^{-1}(U) \subseteq A$ is open, hence there is an open subset $V \subseteq X$ such that $p_A^{-1}(U) = A \cap V$. Now V is saturated with respect to p_X, i.e., $p_X^{-1}(p_X(V)) = V$. By definition of the quotient topology on X/B this shows that $p_X(V) \subseteq X/B$ is open. Now check that $\bar{i}(U) = \bar{i}(A/B) \cap p_X(V)$, and so $\bar{i}(U) \subseteq \bar{i}(A/B)$ is open (in the subspace topology on $\bar{i}(A/B) \subseteq X/B$). This shows that $\bar{i}\colon A/B \to X/B$ is an embedding, and so we can view A/B as a subspace of X/B via $\bar{i}$.

To prove the homeomorphism $(X/B)/(A/B) \cong X/A$ we define mutually inverse maps $f\colon (X/B)/(A/B) \to X/A$ and $g\colon X/A \to (X/B)/(A/B)$ by using the universal property of quotients. The canonical map $X \to X/A$ is constant on $B \subseteq X$, so it descends to a continuous map $X/B \to X/A$. This map is constant on the subspaces A/B, so it descends further to a continuous map $f\colon (X/B)/(A/B) \to X/A$. On the other hand, the composite $X \to X/B \to (X/B)/(X/A)$ is constant on A, hence descends to a continuous map $g\colon X/A \to (X/B)/(A/B)$. It is clear

Date: November 10, 2023.

that f and g are inverses of one another, since both are induced by the identity on X.

Further comments and alternative proofs: There is a useful “pasting lemma” for pushouts that you can prove as an exercise (it only uses the universal property of a pushout): In any category, suppose you are given a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & Y & \longrightarrow & W \end{array}$$

Call the left hand square I, the right hand square II and call the outer square III, that is, III is the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & & \downarrow \\ B & \longrightarrow & W \end{array}$$

Then, *if I and II are pushout squares, then so is III.* Moreover, *if I and III are pushout squares, then so is II.*

Another fact, proved very similarly to (b) above, is the following: Given a pushout square in Top

$$\begin{array}{ccc} A & \xrightarrow{i} & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ Y & \longrightarrow & Y \amalg_A X \end{array}$$

where i is an embedding and f is any map, then also $Y \to Y \amalg_A X$ is an embedding (see Tom Dieck “General Topology”, Prop. 1.8.1 for a proof).

Now consider in the situation of the exercise the commutative diagram

$$\begin{array}{ccccc} B & \longrightarrow & A & \xrightarrow{i} & X \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & A/B & \xrightarrow{\bar{i}} & X/B\,. \end{array}$$

In this case I and III are indeed pushouts (see lecture), so it follows that II is also a pushout. Since $i\colon\ A \to X$ is an embedding, so is $\bar{i}\colon\ A/B \to X/B$. Moreover, we can extend the diagram as follows:

$$\begin{CD}
B @>>> A @>i>> X \\
@VVV @VVV @VVV \\
* @>>> A/B @>\bar{i}>> X/B \\
@. @VVV @VVV \\
@. * @>>> X/A
\end{CD}$$

Call the bottom square IV and call the outer diagram

$$\begin{CD}
A @>i>> X \\
@VVV @VVV \\
* @>>> X/A
\end{CD}$$

V. Since II is a pushout and V is a pushout, IV is also a pushout. But the pushout of $* \leftarrow A/B \xrightarrow{\bar{i}} X/B$ is $(X/B)/(A/B)$, and since pushouts are uniquely determined up to homeomorphism, we must have $(X/B)/(A/B) \cong X/A$.

Exercise 2

(a) If $\alpha\colon G \times X \to X$ is the action map, I'll write $gx := \alpha(g, x)$. Let $p\colon X \to X/G$ be the quotient map. Let $U \subseteq X$ be open. Then

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU\,,$$

where $gU = \{gx \mid x \in U\} \subseteq X$. Since G acts continuously, each g acts through a homeomorphism (in other words, $\alpha(g, -)\colon X \to X$ is a homeomorphism, with inverse $\alpha(g^{-1}, -)$). Since homeomorphisms are open, each gU is open, and hence $p^{-1}(p(U))$ is open, being a union of open sets. By definition of the quotient topology on X/G, this means that $p(U)$ is open. Hence, p is an open map.

(b) Let $H \leq G$ be a normal subgroup. I'll write equivalence classes in X/H as $Hx = \{hx \mid h \in H\}$ and cosets in G/H as Hg. Define an action of G/H on X/H by

$$Hg \cdot Hx = Hgx\,.$$

To see that this is well-defined note that

$$Hhg \cdot Hh'x = Hhgh'x = Hgh'x = Hgh'g^{-1}gx = Hgx\,,$$

where the last equality holds since $gh'g^{-1} \in H$, H being a normal subgroup of G.

To see that the so defined action of G/H on X/H is continuous, we just need to check that each Hg acts continuously on X/H, i.e., that $X/H \xrightarrow{Hg\cdot -} X/H$ is continuous. But for this just note that the composite map $X \xrightarrow{g\cdot -} X \to X/H$ is continuous and H-invariant, and so it descends to a continuous map $X/H \to X/H$; but this map is precisely "action by Hg", i.e., $X/H \xrightarrow{Hg\cdot -} X/H$.

Finally, to prove the homeomorphism $(X/H)/(G/H) \cong X/G$ we construct (similarly to (b) in Exercise 1) maps $f\colon (X/H)/(G/H) \to X/G$ and $f'\colon X/G \to (X/H)/(G/H)$ using the universal property of quotients.

A continuous map out of X/G is defined by defining a continuous G-invariant map out of X (we call a map $F\colon X \to Y$ G-invariant if $F(gx) = F(x)$ for all $x \in X$ and $g \in G$). The composite map $X \to X/H \to (X/H)/(G/H)$ is indeed G-invariant, and so it descends to a map $f'\colon X/G \to (X/H)/(G/H)$. A map in the other direction is constructed similarly. We start with $X \to X/G$ and observe that it is H-invariant. So it descends to a continuous map $X/H \to X/G$. This map is G/H-invariant, and so it descends further to a map $f\colon (X/H)/(G/H) \to X/G$. It is easy to see that f and f' are inverses of one another, because both are essentially induced by the identity map of X.

Exercise 3

(a) To see that $\mathbb{Z} \rtimes \mathbb{Z}$ is generated by $(1,0)$ and $(0,1)$ note that $(1,0)^a = (a,0)$, $(0,1)^b = (0,b)$ and $(a,0)(0,b) = (a,b)$.

Let $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $S(x,y) = (x, y+1)$ and $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ the map defined by $T(x,y)(x+1,-y)$. Define an action of $\mathbb{Z} \rtimes \mathbb{Z}$ on $\mathbb{R}^2$ by

$$(a,b) \cdot (x,y) := S^a T^b (x,y)$$

for $(a,b) \in \mathbb{Z} \rtimes \mathbb{Z}$ and $(x,y) \in \mathbb{R}^2$. We must check that

$$(a,b) \cdot ((a',b') \cdot (x,y)) = ((a,b)(a',b')) \cdot (x,y)$$

for all $(a,b), (a',b') \in \mathbb{Z} \rtimes \mathbb{Z}$ and $(x,y) \in \mathbb{R}^2$. Equivalently, we must check that $S^a T^b S^{a'} T^{b'} = S^{a+(-1)^b a'} T^{b+b'}$. But this follows, because $TS = S^{-1}T$ as one can easily check form the definition of S and T. So we obtain an action of $\mathbb{Z} \rtimes \mathbb{Z}$ on $\mathbb{R}^2$ as desired. It is continuous, because reflection and translation are continuous.

(b) The embedding $\mathbb{Z}^2 \hookrightarrow \mathbb{Z} \rtimes \mathbb{Z}$ defined by $(1,0) \mapsto (1,0)$ and $(0,1) \mapsto (0,2)$ exhibits $\mathbb{Z}^2$ as an index two subgroup of $\mathbb{Z} \rtimes \mathbb{Z}$.

(c) Note that every subgroup of index two is normal. In particular, $\mathbb{Z}^2$ is a normal subgroup of $\mathbb{Z} \rtimes \mathbb{Z}$ and $(\mathbb{Z} \rtimes \mathbb{Z})/\mathbb{Z}^2 \cong C_2$. It follows from Exercise 2 that C_2 acts continuously on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the canonical map $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z} \rtimes \mathbb{Z} = K$ induces a homeomorphism $T^2/C_2 \cong K$.

(d) Recall that a fundamental domain is a subspace such that the orbit of any point intersects that subspace in precisely one point. It is clear that $[0,1)^2$ is a fundamental domain for both $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes \mathbb{Z}$ acting on $\mathbb{R}^2$.

It follows that the inclusion $[0,1]^2 \hookrightarrow \mathbb{R}^2$ induces a surjective map

$$f\colon\ [0,1]^2 \to \mathbb{R}^2/\mathbb{Z}^2\,.$$

On $[0,1]^2$ let $\sim$ be the equivalence relation generated by $(s,0)\sim(s,1)$ and $(0,t)\sim(1,t)$ for all $s,t\in[0,1]$. The map f is invariant under $\sim$, hence it descends to a continuous map

$$\bar{f}\colon\ [0,1]^2/\sim\to \mathbb{R}^2/\mathbb{Z}^2\,.$$

By definition of $\sim$ the map $\bar{f}$ is bijective. To see that $\bar{f}$ is a homeomorphism it remains to show that $\bar{f}$ is open. By the self-indexing trick, it suffices to show that every point $[s,t]\in[0,1]^2/\sim$ has an arbitrarily small open neighbourhood $U_{[s,t]}\subseteq[0,1]^2/\sim$ for which $f(U_{[s,t]})\subseteq\mathbb{R}^2/\mathbb{Z}^2$ is open. If (s,t) is in the interior of $[0,1]^2$ one can take $U_{[s,t]}$ to be the image of a small ϵ-disc. If (s,t) lies on one of the edges of $[0,1]^2$ one can take $U_{[s,t]}$ to be the image of two symmetric ϵ-half-discs around (s,t) and its corresponding mirror point on the opposite edge, respectively. Similarly for the vertices of $[0,1]^2$.

The discussion for the Klein bottle is analogous. The equivalence relation to be put on $[0,1]^2$ is generated by $(s,0)\sim(s,1)$ and $(0,t)\sim(1,1-t)$ for $s,t\in[0,1]$.

Exercise 4

(a) Let $\sim_{S^1}$ be the equivalence relation on $[0,1]$ generated by $0\sim_{S^1}1$. First, one shows that the map $[0,1]\to S^1$ defined by $t\mapsto e^{2\pi it}$ induces a homeomorphism $[0,1]/{\sim_{S^1}}\cong S^1$ (as in Exercise 3 you see that $[0,1]/{\sim_{S^1}}\cong\mathbb{R}/\mathbb{Z}$). Thus, from now on we may take $[0,1]/{\sim_{S^1}}$ as our model for S^1.

The projection $X\times[0,1]\to[0,1]$ is continuous, and its composition with the projection $[0,1]\to[0,1]/{\sim_{S^1}}$ is invariant under $\sim$. Therefore, it descends to a continuous map $T_f\to S^1$.

(b) Let $f\colon\ S^1\to S^1$ be the map $f(z)=z^{-1}$. Under the homeomorphism $S^1\cong[0,1]/\sim_{S^1}$ it corresponds to the map $f\colon\ [0,1]/\sim_{S^1}\to[0,1]/\sim_{S^1}$ given by $f([s])=[1-s]$. Let $\sim_K$ be the equivalence relation on $[0,1]^2$ generated by $(s,0)\sim_K(1-s,1)$ and $(0,t)\sim(1,t)$, so that $[0,1]^2/{\sim_K}$ is the Klein bottle. To construct a map $K\to T_f$ consider the composite map

$$[0,1]\times[0,1]\to([0,1]/{\sim_{S^1}})\times[0,1]\to(([0,1]/{\sim_{S^1}})\times[0,1])/{\sim}=T_f\,.$$

It is invariant under the equivalence relation $\sim_K$ on $[0,1]^2$, and so it descends to a continuous map $K\to T_f$. It is also easily seen to be bijective. It remains to show that it is open. This is a bit tedious, but straightforward. *However, there is one subtlety!* We do not know a-priori that $([0,1]/{\sim_{S^1}})\times[0,1]$ carries the quotient topology with respect to the surjective map $[0,1]\times[0,1]\to([0,1]/{\sim_{S^1}})\times[0,1]$ (it is true though and a consequence of the fact that $[0,1]$ is locally compact). Thus, to check that a subset in $([0,1]/{\sim_{S^1}})\times[0,1]$ is open, you cannot without further justification check if its preimage in $[0,1]^2$ is open.

(c) Tracing carefully through the various constructions above, we find that the map $K = T_f \to S^1$ can be described as the map $[0,1]^2/{\sim_K} \to [0,1]/{\sim_{S^1}}$ sending $[s,t] \mapsto [t]$, and the map $T^2 \to K$ from Exercise 3 (c) can be described as the map $[0,1]^2/{\sim_T} \to [0,1]^2/{\sim_K}$ sending

$$[s,t] \mapsto \begin{cases} [s, 2t] & \text{if } 0 \le t \le \frac{1}{2} \\ [1-s, 2t-1] & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

The composition of the two maps is the map $T^2 \to S^1$ sending

$$[s,t] \mapsto \begin{cases} [2t] & \text{if } 0 \le t \le \frac{1}{2} \\ [2t-1] & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

As we go once around the first circle factor of T^2, the image of this map wraps twice around S^1.

Exercise 5

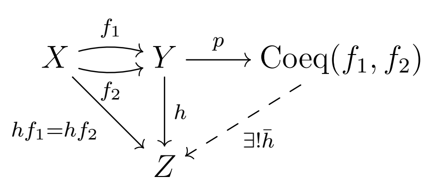
Recall that a coequaliser of morphisms $f_1, f_2\colon X \to Y$ (in any category) is a colimit of the diagram

$$X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y\,.$$

Concretely this means that a coequaliser of $f_1, f_2\colon X \to Y$ is a morphism

$$p\colon Y \to \mathrm{Coeq}(f_1, f_2)$$

such that $pf_1 = pf_2$ (in other words, p coequalises f_1 and f_2) and it is universal with this property in the following sense: Given any morphism $h\colon Y \to Z$ such that $hf_1 = hf_2$, there is a unique $\bar{h}\colon \mathrm{Coeq}(f_1, f_2) \to Z$ such that $\bar{h}p = h$. The following diagram summarises this:



Now consider $f_1, f_2\colon G \times X \to X$ as in the exercise. We claim that the quotient map $p\colon X \to X/G$ is a coequaliser of f_1 and f_2 in Top. Clearly, $pf_1 = pf_2$. So we only need to check the universal property: Let Y be a space and $h\colon X \to Y$ a map such that $hf_1 = hf_2$, i.e., $h(x) = h(gx)$ for all $g \in G$ and $x \in X$. By definition of X/G, there is a unique continuous map $\bar{h}\colon X/G \to Y$ such that $\bar{h}p = h$. So indeed $p\colon X \to X/G$ is a coequaliser of f_1 and f_2.

(We have essentially used this universal property several times in Exercise 2 above).

Exercise 6

Let $\bar{r} \in C_p$ (viewed as an integer $r \in \mathbb{Z}$ modulo p) and let $(z_1, \ldots, z_n) \in S(\mathbb{C}(q_1) \oplus \cdots \oplus \mathbb{C}(q_n))$ and suppose that

$$(e^{2\pi i r q_1/p} z_1, \ldots, e^{2\pi i r q_n/p} z_n) = (z_1, \ldots, z_n)\,.$$

We must show that $\bar{r} = 0$, or in other words that r is divisible by p. Without loss of generality assume that $z_1 \neq 0$. Then $e^{2\pi i r q_1/p} = 1$, hence p divides rq_1. Since q_1 is prime to p, p divides r.