COMMENTS ON SHEET 11

Exercise 4

(a) We show that if f is not surjective, then $\deg(f) = 0$.

If f is not surjective, then there is $p \in S^n$ such that $p \notin f(S^n)$. So f factors as

$$S^n \xrightarrow{f} S^n \backslash \{p\} \hookrightarrow S^n \,.$$

But $S^n\backslash\{p\} \cong D^n \simeq \text{pt}$, and so $H_n(S^n\backslash\{p\}) \cong H_n(\text{pt}) = 0$ $(n \geq 1)$. It follows that $H_n(f)$ factors through 0, hence $H_n(f) = 0$. By definition of degree, $\deg(f) = 0$.

(b) Suppose that f is not surjective. By (a), $f|_{\partial D^n}$ is surjective and so there must be $p \in D^n\backslash\partial D^n$ with $p \notin f(D^n)$. Thus we have a commutative diagram

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f|_{\partial D^n}} & S^{n-1} \\ {\scriptstyle\text{inclusion}}\downarrow & & \simeq\downarrow{\scriptstyle\text{inclusion}} \\ D^n & \xrightarrow{f} & D^n\backslash\{p\} \end{array}$$

in which the right vertical map is a homotopy equivalence. Since $D^n \simeq \text{pt}$, by applying H_{n-1} we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cdot \deg(f|_{\partial D^n})} & \mathbb{Z} \\ \downarrow & & \downarrow\cong \\ 0 & \longrightarrow & \mathbb{Z} \end{array}$$

Since we assumed $\deg(f|_{\partial D^n}) \neq 0$, this is a contradiction.

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