LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2022

Prof. D. Kotschick
Dr. S. Gritschacher

Symplectic Geometry

Sheet 10

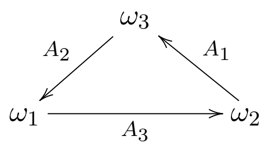
Exercise 1. Let $(\omega, \mathcal{F}, \mathcal{G})$ be a flat Künneth structure on a smooth manifold M. Show that the Künneth structure can always be modified in the neighbourhood of a point $p \in M$ to give a non-flat Künneth structure on M.

Exercise 2. Let M be a smooth manifold, I an almost product structure, and g a pseudo-Riemannian metric such that

$$g(IX, IY) = -g(X, Y) \quad \forall X, Y \in TM\,. \tag{1}$$

1. Show that $\dim(M) = 2n$ for some n, and that g has neutral signature (n, n).
2. Let $F, G \subset TM$ be the eigenbundles of I, and define $\omega(X, Y) := g(IX, Y)$ for $X, Y \in TM$. Show that (ω, F, G) is an almost Künneth structure. Conversely, show that the almost product structure and neutral metric associated to an almost Künneth structure always satisfy (1).
3. Show that the almost Künneth structure (ω, F, G) associated with (g, I) defines a Künneth structure if and only if I commutes with the Levi-Civita connection of g.

Exercise 3. Let M be a smooth manifold with hyperholomorphic symplectic structure given by



Let ∇ denote the Levi-Civita connection of the associated pseudo-Riemannian metric g. Show that A_i commutes with ∇ and that ω_i is ∇-compatible for all $i = 1, 2, 3$. [Hint: Let (ω, J) be one of the (ω_i, A_i) of the hyperholomorphic symplectic structure. To show that $\nabla J = 0$ you can proceed as follows:

1. Prove the following identity for all $X, Y, Z \in \mathfrak{X}(M)$:

   $$2g((\nabla_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ) - g(N(Y, Z), JX)\,,$$

   where $N\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

   $$N(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]\,.$$

2. Assuming that J is integrable, i.e., the $(\pm i)$-eigenbundles of J which are subbundles of $TM \otimes \mathbb{C}$ are closed under the commutator bracket, show that $N(Y, Z) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. For this notice that J and N can be extended $\mathbb{C}$-linearly to sections of $TM \otimes \mathbb{C}$.
3. Show that $\nabla J = 0$.]

(please turn)

Exercise 4. Consider the Lie algebra $\mathfrak{nil}_3 \times \mathbb{R}$ with basis $e_1, \ldots, e_4$ and Lie bracket determined by the single relation $[e_1, e_2] = e_3$. Let $\alpha_1, \ldots, \alpha_4 \in (\mathfrak{nil}_3 \times \mathbb{R})^*$ be the dual basis. We may view $e_1, \ldots, e_4$ as left-invariant vector fields on the corresponding Lie group $\mathrm{Nil}^3 \times \mathbb{R}$ and $\alpha_1, \ldots, \alpha_4$ as the dual left-invariant 1-forms. The commutator relations for $\mathfrak{nil}_3 \times \mathbb{R}$ then translate into

$$d\alpha_3 = \alpha_2 \wedge \alpha_1 \,, \quad d\alpha_i = 0 \quad \forall i \neq 3 \,.$$

Show that the 2-forms

$$\begin{aligned} \alpha &= \alpha_3 \wedge \alpha_2 + \alpha_1 \wedge \alpha_4 \\ \beta &= \alpha_3 \wedge \alpha_1 - \alpha_2 \wedge \alpha_4 \\ \gamma &= \alpha_3 \wedge \alpha_1 + \alpha_2 \wedge \alpha_4 \end{aligned}$$

are left-invariant symplectic forms on $\mathrm{Nil}^3 \times \mathbb{R}$ and define a left-invariant hypersymplectic structure.

Please hand in your solutions in the lecture on 22 July 2022.