Summer term 2022

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Symplectic Geometry

Sheet 8

Exercise 1. Let (M, ω) be an almost symplectic manifold. For $X, Y \in \mathfrak{X}(M)$ let $D(X, Y)$ be the unique vector field satisfying $\iota_{D(X,Y)}\omega = L_X \iota_Y \omega$. Prove that for vector fields X and Y satisfying $\omega(X, Y) = 0$ we have $D(fX, Y) = fD(X, Y)$.

Exercise 2. Let (M, g) be a Riemannian manifold. Let $J \in \Gamma(\mathrm{End}(TM))$ be an almost complex structure on M that is compatible with g in the sense that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. We call (M, g, J) an *almost Hermitian manifold.*

1. Define a two-form ω on M by $\omega(X, Y) := g(JX, Y)$ for $X, Y \in \mathfrak{X}(M)$. Show that ω is non-degenerate and J-invariant, that is $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$.
2. Suppose that $\mathcal{F}$ is a Lagrangian foliation on (M, ω), and that the Levi-Civita connection ∇^{LC} of g preserves $T\mathcal{F}$. Prove that $J(T\mathcal{F})$ is a Lagrangian complement to $T\mathcal{F}$ that is also preserved by ∇^{LC}. Conclude that $J(T\mathcal{F})$ is integrable to a Lagrangian foliation $\mathcal{G}$.

Exercise 3. Let M be a smooth manifold and ∇ an affine connection on M.

1. Show that ∇ induces a covariant derivative on $\mathrm{End}(TM)$, by the formula[1]

$$(\nabla_X A)(Y) := \nabla_X(A(Y)) - A(\nabla_X(Y)), \quad A \in \Gamma(\mathrm{End}(TM)),\ X, Y \in \mathfrak{X}(M).$$

   Thus ∇ commutes with A precisely if A is ∇-parallel: $\nabla A = 0$.

2. Let (M, g, J) be an almost Hermitian manifold, and assume further that $\nabla^{\mathrm{LC}} J = 0$. We call (M, g, J) a *Kähler manifold.* Let $(\omega, \mathcal{F}, \mathcal{G})$ be as in Exercise 2. Show that in this situation the Levi-Civita connection ∇^{LC} is the Künneth connection of $(\omega, \mathcal{F}, \mathcal{G})$.

Exercise 4. Let (ω_1, F_1, G_1) and (ω_2, F_2, G_2) be almost Künneth structures on M_1 respectively M_2. Show that $(\omega_1 \pm \omega_2, F_1 \oplus F_2, G_1 \oplus G_2)$ are almost Künneth structures on $M_1 \times M_2$, whose Künneth connections are the direct sum connections of the Künneth connections on the two factors.

Please hand in your solutions in the lecture on 8 July 2022.

[1] Note that we can view a section $A \in \Gamma(\mathrm{End}(TM))$ as a $C^\infty(M)$-linear map $\Gamma(TM) \to \Gamma(TM)$ and vice versa.