Summer term 2022

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Symplectic Geometry

Sheet 7

Exercise 1. Let G be a Lie group acting smoothly on a symplectic manifold (M, ω). Prove that the action by G is Hamiltonian if and only if there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$, $X \mapsto \mu^X$ such that μ^X is a Hamiltonian function for $X^\sharp$. (The Lie bracket on $C^\infty(M)$ is the Poisson bracket $\{f, g\} = \omega(X_f, X_g)$.)

Exercise 2. Suppose that G acts smoothly on a connected symplectic manifold (M, ω), and let $\mathfrak{g} \to C^\infty(M)$, $X \mapsto \mu^X$ be a linear map such that μ^X is a Hamiltonian function for $X^\sharp$.

1. Prove that there is a bilinear form $\tau\colon\ \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ satisfying

$$\tau(X, Y) = \{\mu^X, \mu^Y\} - \mu^{[X,Y]}$$

for all $X, Y \in \mathfrak{g}$. Verify the identity $\tau([X, Y], Z) + \tau([Z, X], Y) + \tau([Y, Z], X) = 0$.

2. Now assume that there is a linear map $\sigma\colon\ \mathfrak{g} \to \mathbb{R}$ such that $\tau(X, Y) = \sigma([X, Y])$. Show that the action by G is Hamiltonian. [Hint: Modify μ using σ]

Exercise 3. View an element $\varphi \in SL_2(\mathbb{Z})$ as a linear diffeomorphism of T^2. Assume that φ preserves a 1-dimensional foliation $\mathcal{F}$. Prove that the 4-manifold $N = M(\varphi) \times S^1$, where $M(\varphi)$ denotes the mapping torus associated with φ, carries an induced Künneth structure.

Exercise 4. Let $\varphi\colon\ M \to M$ be a diffeomorphism of a connected smooth manifold M. Let $M(\varphi)$ be the mapping torus associated with φ.

1. Prove that $H^1_{\mathrm{dR}}(M(\varphi)) \cong \mathbb{R} \oplus H^1_{\mathrm{dR}}(M)^\varphi$, where $H^1_{\mathrm{dR}}(M)^\varphi$ denotes the subspace of $H^1_{\mathrm{dR}}(M)$ fixed by φ^*. [Hint: You can use the Mayer-Vietoris sequence.]

2. Find an example of a 4-manifold, equipped with a Künneth structure, of the form $M(\varphi) \times S^1$ whose first Betti number ($= \dim H^1_{\mathrm{dR}}$) equals 3.

Please hand in your solutions in the lecture on 1 July 2022.