Summer term 2022

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Symplectic Geometry

Sheet 6

Exercise 1. Let (M, ω) be a symplectic manifold. Assume that ω is exact and choose a 1-form λ such that $\omega = -d\lambda$. An action of a Lie group G on M is called *exact* if $\psi_g^*(\lambda) = \lambda$ for every $g \in G$.

1. Prove that every exact action is Hamiltonian with $\mu^X = \iota_{X^\sharp}\lambda$ for $X \in \mathfrak{g}$.

Now let L be any manifold.

2. Prove that an action of G on L lifts to a Hamiltonian action of G on $(T^*L, \omega_{\mathrm{can}})$.

Exercise 2. Consider the natural action of $U(n)$ on $(\mathbb{C}^n, \omega_0)$, where $\omega_0 = \frac{i}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$. We identify the Lie algebra

$$\mathfrak{u}(n) = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid X^\dagger = -X\}$$

with its dual $\mathfrak{u}(n)^*$ via the inner product $(X, Y) = \mathrm{tr}(X^\dagger Y)$. Prove that the action is Hamiltonian with moment map $\mu\colon \mathbb{C}^n \to \mathfrak{u}(n)$ given by

$$\mu(z) = \frac{i}{2} z z^\dagger\,.$$

[Hint: You can use Exercise 1.]

Exercise 3. Suppose that a Lie group G acts in a Hamiltonian way on two symplectic manifolds (M_i, ω_i) with moment maps $\mu_i\colon M_i \to \mathfrak{g}^*$, $i = 1, 2$.

1. Prove that the diagonal action of G on $M_1 \times M_2$ (with the product symplectic structure) is Hamiltonian with moment map $\mu\colon M_1 \times M_2 \to \mathfrak{g}^*$ given by $\mu(p, q) = \mu_1(p) + \mu_2(q)$.

Now consider the natural action of $U(n)$ on $((\mathbb{C}^n)^k, \omega_0)$.

2. Prove that the action is Hamiltonian with moment map $\mu\colon (\mathbb{C}^n)^k \to \mathfrak{u}(n)$ given by $\mu(A) = \frac{i}{2}AA^\dagger + \frac{id}{2i}$. [The constant $\frac{id}{2i}$ is only a matter of convenience for part (3).]
3. Prove that if $n \leq k$, then $\mu^{-1}(0)/U(n) = \mathrm{Gr}_n(\mathbb{C}^k)$ is the Grassmannian of n-dimensional planes in $\mathbb{C}^k$.

Exercise 4. Consider the Hopf fibration $\pi\colon S^3 \to S^2$, which is a principal S^1-bundle. Show that the foliation of S^3 with leaves the fibers of π does not admit a complementary foliation. [Hint: If $\mathcal{F}$ is a foliation transverse to the fibers, then show that for every leaf L of $\mathcal{F}$ the map $\pi|_L\colon L \to S^2$ is a covering. Then use the fact that a connected covering of S^2 is again S^2.]

Please send your solutions per email until 17 June 2022.