LMU LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2022

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Symplectic Geometry

Sheet 5

Exercise 1. Let (M, ω) be a symplectic manifold and $\mathcal{F}$ a Lagrangian foliation. Recall that the Bott connection ∇ on $T\mathcal{F}$ is determined by the following identity:

$$\omega(\nabla_X Y, Z) = L_X(\omega(Y, Z)) + \omega([X, Z], Y)\,, \quad X, Y \in \Gamma(T\mathcal{F}),\ Z \in \mathfrak{X}(M)\,.$$

Give a direct proof of the fact that ∇ is flat on every leaf of $\mathcal{F}$.

Exercise 2. Let M be a smooth manifold.

1. For $X, Y \in \mathfrak{X}(M)$ and α a smooth form on M prove that

$$\iota_{[X,Y]}\alpha = L_X \iota_Y \alpha - \iota_Y L_X \alpha\,.$$

2. Let (M, ω) be a symplectic manifold. For $H \in C^\infty(M)$ define the Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ by the identity $\iota_{X_H}\omega = dH$. For $H, G \in C^\infty(M)$ prove that

$$[X_H, X_G] = X_{\omega(X_G, X_H)}\,.$$

Exercise 3. Let (M, ω) be a symplectic manifold and $Q \subset M$ a coisotropic submanifold. Recall that this means that at every $p \in Q$ the tangent space $T_pQ \subset T_pM$ satisfies $(T_pQ)^\perp \subset T_pQ$. Show that the distribution $TQ^\perp \subset TQ$ is integrable. You can proceed as follows:

1. Suppose that Q has codimension k. Let $p \in Q$ and let $H_1, \ldots, H_k \in C^\infty(U)$ be functions which define Q in a neighbourhood $U \subset M$ of p.[1] Prove that the vector fields X_{H_i} span $T_xQ^\perp$ at every $x \in Q \cap U$.
2. Use the Frobenius theorem to prove that $TQ^\perp$ is integrable.

Exercise 4. Let M be a smooth manifold and $\pi\colon T^*M \to M$ its cotangent bundle. Prove that there is a bundle isomorphism $\Phi\colon \pi^*(TM \oplus T^*M) \to T(T^*M)$ which identifies $\pi^*(T^*M)$ with the vertical tangent bundle, and such that $D\pi \circ \Phi$ restricts to the identity on the summand $\pi^*(TM)$ and Φ pulls back the canonical symplectic form ω_{can} on T^*M to the standard form on $\pi^*(TM) \oplus \pi^*(T^*M)$.

Please hand in your solutions in the lecture on Friday, 10 June 2022.

[1] This means that the differentials $dH_1, \ldots, dH_k$ are linearly independent and $Q \cap U$ is the set of common zeros of $H_1, \ldots, H_k$.