Summer term 2022

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Symplectic Geometry

Sheet 1

Exercise 1. Let (V, ω) be a symplectic vector space and $W \subset V$ a linear subspace. Prove that the quotient space $W/W \cap W^{\perp}$ carries a natural symplectic structure.

Exercise 2. Let (V, ω) be a symplectic vector space. We call a subspace $W \subset V$ *coisotropic* if $W^{\perp} \subset W$.

1. Show that any subspace of V of codimension one is coisotropic.
2. Let $W \subset V$ be coisotropic and let $U \subset V$ be Lagrangian. Show that the quotient space
$$((U \cap W) + W^{\perp})/W^{\perp}$$
is a Lagrangian subspace of $W/W^{\perp}$.

Exercise 3. Let W be a finite dimensional real vector space. Following the lecture, we consider $V = W \oplus W^*$ equipped with the symplectic form
$$\omega((v, \lambda), (w, \mu)) = \lambda(w) - \mu(v)\,.$$
A linear map $\alpha\colon W^* \to W$ is called *self-adjoint* with respect to the natural pairing between W and W^* if
$$\lambda(\alpha(\mu)) = \mu(\alpha(\lambda))$$
for all $\mu, \lambda \in W^*$.

Let $A \subset V$ be a linear subspace complementary to W, defined as the graph of a linear map $\alpha\colon W^* \to W$. Show that A is Lagrangian if and only if α is self-adjoint. Conclude that the space of Lagrangian complements to W is a vector space and compute its dimension.

Exercise 4. Let (V, ω) be a symplectic vector space and $W \subset V$ a linear subspace.

1. Show that W has a basis $u_1, \ldots, u_k, v_1, \ldots, v_k, w_1, \ldots, w_p$ such that $\omega(u_i, v_j) = \delta_{ij}$ and $\omega(a, b) = 0$ for any other pair of basis vectors a, b.
2. Show that if W is isotropic, coisotropic, or symplectic, then any basis as above can be extended to a symplectic basis for (V, ω).

Please hand in your solutions in the lecture on Friday, 6 May 2022.