LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

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# Functional Analysis
## Tutorial 11

**Problem 1**. Let $\mathcal{H}$ be a Hilbert space, and let $T : \mathcal{H} \to \mathcal{H}$ be a symmetric operator, i.e. $\langle x, Ty \rangle = \langle Tx, y \rangle$ for all $x, y \in \mathcal{H}$.

$(i)$ Prove that $T$ is bounded, by using the Uniform Boundedness Principle.
[*Hint:* Prove first that $\{f_y \in \mathcal{H}' \mid f_y(x) := \langle Ty, x \rangle, \|y\| = 1\}$ is pointwise bounded.]

$(ii)$ Prove that $T$ is bounded, by using the Closed Graph Theorem.

**Problem 2**. Let $X$ be the Banach space of continuous real-valued functions $C([0,1], \mathbb{R})$, equipped with $\|\cdot\|_\infty$, and let $Y$ be the normed linear space $C([0,1], \mathbb{R})$, equipped with $\|\cdot\|_1$, where $\|f\|_1 = \int_0^1 |f(x)|\, dx$.

$(i)$ Prove that the identity operator $\mathrm{id}_{\mathrm{X,Y}} : X \to Y, f \mapsto f$ is bounded.

$(ii)$ Prove that the identity operator $\mathrm{id}_{\mathrm{Y,X}} : Y \to X, f \mapsto f$ is *not* bounded.

$(iii)$ Prove that $\mathrm{id}_{X,Y}$ is not an open map.

$(iv)$ Prove that $Y$ is not complete, by using

- $(a)$ the Open Mapping Theorem.
- $(b)$ the Inverse Mapping Theorem.
- $(c)$ Corollary 4.8 of the Inverse Mapping Theorem.
- $(d)$ Corollary 4.3 of Baire's Category Theorem.
  [*Hint:* Prove that, for each $n \in \mathbb{N}$, the set $B_n := \{f \in C([0,1], \mathbb{R}) \mid \|f\|_\infty \leqslant n\}$ is nowhere dense in $Y$ and conclude that $Y$ is meagre (i.e. that it is a countable union of nowhere dense sets.]