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# Functional Analysis
## Tutorial 10

**Problem 1**. Which of the following operators are compact? Prove your claims.

(*i*) $T_1 : X \to X, x \mapsto x$, where $X$ is an infinite-dimensional Banach space.

(*ii*) $T_2 : \ell^p \to \ell^q, x \mapsto x$, where $1 \leqslant p < q \leqslant \infty$.

(*iii*) $T_3 : X \to X$ such that $T_3^{-1}$ is a bounded linear map, where $X$ is an infinite-dimensional Banach space.

(*iv*) $T_4 : C([0,1]) \to C([0,1]), f \mapsto Tf$, $Tf(x) := f(0) + xf(1)$ for all $x \in [0,1]$.

**Problem 2**. Let $a, b > 0$ and

$$E := \left\{ f \in C^1([0,1]) \,\middle|\, |f(0)| \leqslant a, \int_0^1 |f'(x)|^2 dx \leqslant b^2 \right\}.$$

Prove that $E$ is relatively compact in $C([0,1])$.

**Problem 3**. Let $\mathcal{H}$ be a separable Hilbert space, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, and let $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ be an orthonormal system such that

$$\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty .$$

(*i*) Let $N \in \mathbb{N}$ be such that $\sum_{n>N} \|\varphi_n - \psi_n\|^2 < 1$, and let $V_N := (\{\psi_n\}_{n>N})^\perp$. Prove that the map

$$P : V_N \to \operatorname{span}\{\varphi_n\}_{n=1}^N, \xi \mapsto \sum_{n=1}^{N} \langle \varphi_n, \xi \rangle \, \varphi_n$$

is injective.

(*ii*) Prove that, for $N$ and $V_N$ as in (*i*), we have $\dim(V_N) = N$, and conclude that $V_N = \operatorname{span}\{\psi_n\}_{n=1}^N$.

(*iii*) Prove that $\{\psi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$.