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Functional Analysis
Tutorial 6

Problem 1. Let (X,d) be a metric space. Recall, that for a subset $A \subset X$, the *distance* of a point $x \in X$ to A is defined by $\operatorname{dist}(x,A) := \inf_{y\in A} d(x,y)$. Prove that for a subset $A \subset X$ the following statements are equivalent:

(a) A is closed.

(b) $d(x,A) = 0$ if and only if $x \in A$.

Problem 2. Let X be a normed space, and let $E, F \subset X$ be non-empty. Prove:

(i) If E is open then $E+F$ is open.

(ii) If E is closed and F is compact, then $E+F$ is closed.

(iii) For $E+F$ to be closed, it is generally not sufficient that E and F are closed.

(iv) If E and F are compact, then $E+F$ is compact.

Problem 3. For a metric space X, $r>0$, and $x \in X$, let $B_r(x)$ denote the open ball and let $K_r(x)$ denote the closed ball in X (compare Tutorial 3, Problem 1).

(i) Let (X,d) be a metric space. Prove that the following statements are equivalent:

- (a) For all $x \in X$ and $r>0$, we have $\overline{B_r(x)} = K_r(x)$.
- (b) For all $x, y \in X$ with $x \neq y$ and for all $\varepsilon > 0$, there exists $z \in X$ such that $d(y,z) < \varepsilon$ and $d(x,z) < d(x,y)$.

(ii) Let $(X, \|\cdot\|)$ be a normed space. Prove that $\overline{B_r(x)} = K_r(x)$ for all $r>0$ and $x \in X$.

Problem 4. Prove that

$$c_{00} := \left\{ (x_n)_{n\in\mathbb{N}} \,\middle|\, x_n \in \mathbb{C} \textit{ and } x_n = 0 \textit{ for all but finitely many } n \in \mathbb{N} \right\}$$

is dense in $(\ell^p, \|\cdot\|_p)$ for $1 \leqslant p < \infty$.