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Functional Analysis
Tutorial 4

Problem 1 (Equivalence of metrics). Two metrics on a set X are called *(topologically) equivalent*, if they generate the same topology on X. Prove:

(*i*) Two metrics d and d' on a set X are topologically equivalent, if and only if the convergent sequences in (X,d) are the same as the convergent sequences in (X,d').

(*ii*) If, for two metrics d and d' on a set X, there exist constants $c, C > 0$ such that

$$c\,d(x,y) \leqslant d'(x,y) \leqslant C\,d(x,y) \quad \text{for all } x,y \in X\,, \tag{$*$}$$

then they are topologically equivalent[1].

(*iii*) A metric d on a set X is topologically equivalent to the metrics $\min\{1,d\}$ and $\frac{d}{1+d}$ (hence every metric is topologically equivalent to a bounded metric), but not strongly equivalent.

Consider the metrics d_1 and d_∞ on $\mathbb{R}^d$ and $C([0,1])$, defined by

$d_1(x,y) := \sum_{j=1}^n |x_j - y_j|$, $d_\infty(x,y) := \max_{j=1,\dots,n} |x_j - y_j|$ for all $x,y \in \mathbb{R}^n$,

$d_1(f,g) := \int_0^1 |f(x) - g(x)|\,dx$, $d_\infty(f,g) := \max_{x\in[0,1]} |f(x) - g(x)|$ for all $f,g \in C([0,1])$.

(*iv*) Prove that d_1 and d_∞ on $\mathbb{R}^d$ are strongly equivalent.

(*v*) Prove that d_1 and d_∞ on $C([0,1])$ are not equivalent.

Problem 2.

(*i*) Find a (non-complete) metric space (X,d) and a contraction $\phi : X \to X$ such that ϕ has no fixed points.

(*ii*) Prove that every compact metric space is bounded.

(*iii*) Let $\mathcal{B}$ be a base for the topology of a topologcal space $(X,\mathcal{T})$, and assume that every cover of X by elements from $\mathcal{B}$ has a finite subcover. Prove that X is compact.

(*iv*) Prove that a topological space $(X,\mathcal{T})$ is compact, if and only if any family $\mathcal{F}$ of closed subsets of X, such that $\bigcap_{n=1}^N F_n \neq \emptyset$ for any finite subfamily $\{F_n\}_{n=1}^N \subset \mathcal{F}$, satisfies $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.

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[1] Two metrics d and d' that satisfy $(*)$ are called *strongly* (or *uniformly*) *equivalent*.

Problem 3. Let (X, d) and (Y, d') be metric spaces. Prove:

(i) For any $x_0 \in X$, the function $f : X \to \mathbb{R}, x \mapsto d(x, x_0)$ is uniformly continuous.

(ii) If X is compact, then each continuous function $f : X \to Y$ is uniformly continuous.