Prof. M. Fraas, PhD
A. Groh, S. Gottwald

Summer term 2015
June 29, 2015

Functional Analysis
Exercise Sheet 11

Banach-Alaoglu, Baire Category Theorem, Uniform Boundedness

- *First version deadline:* **July 6** (13:30). *Final hand in deadline:* **July 15** (13:30)

We have often stressed the importance of weakly compact sets. The result of the following exercise provides a whole family of such sets.

Exercise 1 (5 points). *Prove that a norm-closed convex bounded subset of a reflexive Banach space is weakly compact.*

Exercise 2 (5 points). *Let $X := C([-1,1])$ be equipped with $\|\cdot\|_1$, and for each $n \in \mathbb{N}$ let $\delta_n : C([-1,1]) \to \mathbb{C}$ be given by*

$$\delta_n(f) = \frac{n}{2} \int_{-1/n}^{1/n} f(x)\, dx\,.$$

Prove:

(i) $\delta_n \in X^*$ *for all* $n \in \mathbb{N}$, *and* $\|\delta_n\| \to \infty$ *as* $n \to \infty$,

(ii) $\delta_\infty(f) := \lim_{n\to\infty} \delta_n(f)$ *exists for all* $f \in X$, *in particular* $\sup_n |\delta_n(f)| < \infty$ *for all* $f \in X$. *How is this consistent with the Uniform Boundedness Principle?*

(iii) δ_∞ *is not a bounded functional.*

Exercise 3 (5 points). *Prove:*

(i) Any closed proper subspace of a normed space X is nowhere dense.

(ii) Any Hamel basis in an infinite dimensional Banach space X is uncountable.

(iii) The space $\mathcal{P}$ of polynomials cannot be equipped with a complete norm, i.e. with a norm $\|\cdot\|$ that makes $(\mathcal{P}, \|\cdot\|)$ complete.

The open mapping principle guarantees that a continuous linear bijection between Banach spaces has a continuous inverse. In the following exercise we give some counterexamples that show that the linearity assumption and the right topology are important. The failure of a continuous bijection to be a homeomorphism is often connected with a topological obstruction or shifting to/from infinity.

Exercise 4 (5 points). *Prove that the following maps $\phi : X \to Y$ between topological spaces X, Y are continuous bijections that are not homeomorphisms:*

(i) $X = [0, 2\pi)$, $Y = S_1 := \{x \in \mathbb{R}^2 : |x_1|^2 + |x_2|^2 = 1\}$, $\phi(x) := (\cos(x), \sin(x))$

(ii) $X = Y$ *is the space of bounded sequences* $(x_n)_{n\in\mathbb{Z}}$, *equipped with the topology in which* $U \subset X$ *is open either if* $U = X$, $U = \emptyset$, *or if for all* $x \in U$ *we have* $x_n = 0$ *for* $n < 0$, *and* $\phi : X \to X$ *is given by* $(\phi(x))_n = x_{n+1}$.

We saw in the lecture that there exist discontinuous linear maps on any infinite dimensional Banach space, although no such map can be explicitly constructed. Existence of a discontinuous linear map is rather obscure and can be ruled out by imposing further conditions. An example is a Hermitian condition that plays a very important role in physics.

Exercise 5 (5 points). *Let* $\mathcal{H}$ *be a Hilbert space. Prove that an everywhere defined linear map* $T : \mathcal{H} \to \mathcal{H}$ *is bounded if it is symmetric, i.e. if* $(x, Ty) = (Tx, y)$ *for all* $x, y \in \mathcal{H}$.

Exercise 6 (5 points). *Let* X *be an infinite-dimensional Banach space. Prove*

(i) *The weak topology on* X *is not first countable.*

(ii) *The weak topology on* X *is not metrizable, i.e. there is no metric* d *on* X *that generates the weak topology.*

For general informations please visit http://www.math.lmu.de/~gottwald/15FA/