Functional Analysis

E17 [7 points]. Let X be a normed space with $\dim X = \infty$ and let $B \subset X$ be a Hamel basis of X.

(*i*) Let $L \subset X$ be a finite-dimensional subspace. Prove that L is nowhere dense in X.

(*ii*) Show that if X is complete, then B is uncountable.

(*iii*) Let $\mathcal{P}$ be the linear space of all real-valued polynomials on $\mathbb{R}$ and for $p \in \mathcal{P}$, given by $p(t) = \sum_{k=0}^{n} a_k t^k$, define $\|p\| := \sum_{k=0}^{n} |a_k|$. Prove that $(\mathcal{P}, \|\cdot\|)$ is a normed space and argue whether or not it is complete.

E18 [3 points]. Find an example for two norms on the same vector space that are not equivalent (and prove your claim).

E19 [6 points]. For $n \in \mathbb{N}_0$ and a non-empty open intervall $I \subset \mathbb{R}$ we define

$$C_b^n(I) := \left\{ f \in C(I) \,\middle|\, f \text{ } n\text{-times continuously differentiable and } \|f\|_{n,\infty} < \infty \right\}$$

where $\|f\|_{n,\infty} := \sum_{k=0}^{n} \|f^{(k)}\|_\infty$ and $\|f\|_\infty := \sup_{x \in I} |f(x)|$. Show:

(*i*) $(C_b^1(I), \|\cdot\|_{1,\infty})$ is a Banach space.

(*ii*) $(C_b^n(I), \|\cdot\|_\infty)$ is not complete for $n \geqslant 1$.

E20 [8 points]. Let $X := \{x \in \ell^1 : |||x||| < \infty\}$, where $|||x||| := \sum_{j=1}^{\infty} j|x_j|$. Moreover, let J be the embedding of $(X, |||\cdot|||)$ in $(\ell^1, \|\cdot\|_1)$, i.e. $J : (X, |||\cdot|||) \to (\ell^1, \|\cdot\|_1), x \mapsto x$. Prove the following statements:

(*i*) $(X, |||\cdot|||)$ is a Banach space, whereas $(X, \|\cdot\|_1)$ is not.

(*ii*) J is a bounded linear map (hence X *is continuously embedded in* ℓ^1).

(*iii*) The image of $\bar{B}_1^{|||\cdot|||}(0) = \{x \in X : |||x||| \leqslant 1\}$ under J is compact (as we shall see later, this means that *the embedding J is a compact operator*).

Please hand in your solutions until next **Wednesday (14.05.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/