MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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Summer term 2014
Sheet 1
09.04.2014

Functional Analysis

E1 [6 points]. Let $(Y, \mathcal{T})$ be a Hausdorff space.

(i) Prove that the limit of convergent sequences in Y is unique.

(ii) Show that for every $a \in Y$ the singleton $\{a\}$ is closed.

(iii) Let $(X, \mathcal{T}')$ be another topological space, and let $f : X \to Y$ be continuous. Prove that the graph of f, $\Gamma(f) := \{(x, f(x)) \mid x \in X\}$, is closed in $X \times Y$ with respect to the product topology.

E2 [6 points]. Let (X, d) be a metric space. Let $x \in X$, $\varepsilon > 0$ and let $B_\varepsilon(x)$ denote the *open ball* of radius ε centered at x, i.e. $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$.

(i) Show that the *closed ball* $\bar{B}_\varepsilon(x) := \{y \in X \mid d(x, y) \leqslant \varepsilon\}$ is closed.

(ii) Prove that $\overline{B_\varepsilon(x)} \subseteq \bar{B}_\varepsilon(x)$.

(iii) Give an example for a metric space where, in general, $\overline{B_\varepsilon(x)} \neq \bar{B}_\varepsilon(x)$.

E3 [6 points]. Let $X = Y := \mathbb{R}$ and consider the two topological spaces $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$, where $\mathcal{T}_2$ is the standard topology on $\mathbb{R}$ induced by the Euclidean metric and

$$\mathcal{T}_1 := \{\emptyset, \mathbb{R}\} \cup \{\mathbb{R} \setminus A \mid A \subseteq \mathbb{R} \text{ is countable}\}.$$

Prove the following:

(i) The pair $(X, \mathcal{T}_1)$ is in fact a topological space.

(ii) Every mapping $F : X \to Y$ is sequentially continuous.

(iii) The mapping $G : X \to Y$, $x \mapsto x$ is not continuous.

E4 [6 Points].

(i) Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on a set X, and let $\mathcal{B}$ and $\mathcal{B}'$ be bases of $\mathcal{T}$ and $\mathcal{T}'$ respectively. Show that $\mathcal{T} \subset \mathcal{T}'$ (i.e. $\mathcal{T}'$ is *finer* than $\mathcal{T}$) if and only if for each $x \in X$ and each $A \in \mathcal{B}$ with $x \in A$, there is $A' \in \mathcal{B}'$ such that $x \in A'$ and $A' \subseteq A$.

(ii) Show that $d'(x, y) := |e^x - e^y|$ defines a metric on $\mathbb{R}$.

(iii) Prove that d' induces the same topology as the standard metric on $\mathbb{R}$.

(iv) Show that $(\mathbb{R}, d')$ is not complete. [*Hence completeness is not a topological property.*]

Please hand in your solutions until next **Wednesday (16.04.2014)** *at* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/