ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 3

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Exercise 1. [15 points]

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function vanishing at infinity, we denote by f^* the symmetric-decreasing rearrangement of a f. Prove that

1. $f^*(x)$ is radially symmetric and nonincreasing, i.e.,

$$f^*(x) = f^*(y) \text{ if } |x| = |y|,$$

   and

$$f^*(x) \geq f^*(y) \text{ if } |x| \leq |y|,$$

2. $f^*(x)$ is a lower semicontinuous function since the sets $\{x \,|\, f^*(x) > t\}$ are open for all $t > 0$.
3. The level sets of f^* are simply the rearrangements of the level sets of $|f|$, i.e.,

$$\{x \,|\, f^*(x) > t\} = \{x \,|\, |f(x)| > t\}^*$$

Exercise 2. [15 points]

Let $\mathcal{H}$ be an Hilbert space and let $M \subset \mathcal{H}$ be a subspace.

1. Show that the orthogonal complement of M, $M^\perp$, is closed.
2. If $M \subset \mathcal{H}$ is closed, prove that every $x \in \mathcal{H}$ can be uniquely written as $x = z + w$, where $z \in M$ and $w \in M^\perp$.
3. Prove that $\overline{M}^\perp = M^\perp$.

Exercise 3. [10 points]

Let $\mathcal{H}$ be an Hilbert space and let $\mathcal{H}^*$ be the dual space of $\mathcal{H}$. Prove that for each $T \in \mathcal{H}^*$ there is a unique $y_T \in \mathcal{H}$ such that $T(x) = (y_T, x)$ for all $x \in \mathcal{H}$. In addition $\|y_T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}^*}$.