Remarks on Orthocenters, Pappus' Theorem and Butterfly Theorems

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To the memory of my colleague Günter Pickert

Abstract. We present a generalization of the notion of the orthocenter of a triangle and of Pappus' theorem. Both subjects were discussed with Pickert in the last year of his life. Furthermore we add a projective Butterfly Theorem which covers all known affine cases.

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Keywords. orthocenter, Pappus' Theorem, Butterfly Theorem.

Günter Pickert was an excellent mathematician obtaining very deep results. But he was also interested in problems of elementary mathematics in particular elementary geometry, see his last publications for which the author worked as coauthor [4].

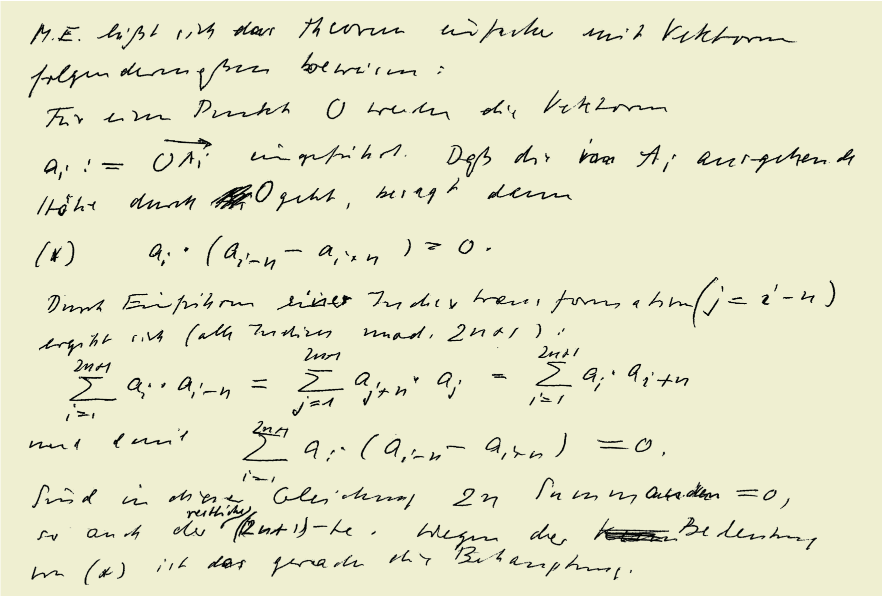
1. Orthocenter for $(2n+1)$-gons

Let $\mathcal{P}$ be a plane polygon with $2n+1$ vertices, $n > 0$. The vertices A_i are indexed by $-n$, $-n+1$, ..., -1, 0, 1, ..., $n-1$, n considered as elements of the factor group $\mathbb{Z}_{2n+1} = \mathbb{Z}/(2n+1)\mathbb{Z}$.

Definition 1.1.

1. Given a vertex A_i, the line connecting the vertices A_{i-n}, A_{i+n} is called the *opposite side* of the vertex A_i.
2. The lines dropped from a vertex to its opposite side are called *altitudes* of the polygon $\mathcal{P}$.

Theorem 1.2. *If $2n$ altitudes of the polygon $\mathcal{P}$ concur then the remaining altitude also passes through the point of concurrence, called the* orthocenter *of the polygon $\mathcal{P}$.*



M.E. läßt sich das Theorem einfacher mit Vektoren folgendermaßen beweisen:

Für einen Punkt O werden die Vektoren $a_i := \overrightarrow{OA_i}$ eingeführt. Daß die von A_i ausgehende Höhe durch O geht, besagt dann

$$(*) \quad a_i \cdot (a_{i-n} - a_{i+n}) = 0.$$

Durch Einführen einer Indextransformation $(j = i - n)$ ergibt sich (alle Indizes mod. $2n+1$):

$$\sum_{i=1}^{2n+1} a_i \cdot a_{i-n} = \sum_{j=1}^{2n+1} a_{j+n} \cdot a_j = \sum_{i=1}^{2n+1} a_i \cdot a_{i+n}$$

und damit $\sum_{i=1}^{2n+1} a_i \cdot (a_{i-n} - a_{i+n}) = 0$.

Sind in dieser Gleichung $2n$ Summanden $=0$, so auch der restliche $(2n+1)$-te. Wegen der Bedeutung von $(*)$ ist das gerade die Behauptung.

Figure 1. Letter by Pickert

The author presented this theorem with a proof to Pickert in a letter dated by March 21, 2014. He answered on March 24, 2014 with a much simpler proof (Figure 1).
Although the handwriting of 97 years old Günter Pickert is very readable we provide a translation for the convenience of the reader.

Proof. "In my opinion the theorem can be proved more simply by means of vectors in the following way.

For a point O one introduces the vectors $a_i := \overrightarrow{OA_i}$. That the altitude dropped from A_i passes through the point O means

$$a_i \cdot (a_{i-n} - a_{i+n}) = 0. \tag{1.1}$$

Inserting an index transformation $j = i - n$ one obtains[1]

$$\sum_{i=1}^{2n+1} a_i \cdot a_{i-n} = \sum_{j=1}^{2n+1} a_{j+n} \cdot a_j = \sum_{i=1}^{2n+1} a_i \cdot a_{i+n} \tag{1.2}$$

which implies

$$\sum_{i=1}^{2n+1} a_i \cdot (a_{i-n} - a_{i+n}) = 0. \tag{1.3}$$

Now, if $2n$ summands in this sum vanish the remaining one does so." □

[1]Note that Pickert labels the vertices different from our use by 1, 2, …, $2n+1$.

Following the advice of Chris Fisher the author proposed the proof of this theorem as a problem to the Canadian journal *Crux Mathematicorum.* There Pickert's proof appeared in print [5].

Remark 1.3. The theorem can be seen as a closure theorem in the following sense. Let $2n+1$ mutually different, but concurrent lines h_i, $-n < i < n$, be given such that none of them is orthogonal to another one. Choose a point A_{-n} on the line h_{-n} different from the point O of concurrence and drop the perpendicular from this point to the line h_1. Since the line h_{-n} is not orthogonal to the line h_1 this perpendicular does not pass through the point O and since the line h_1 is not orthogonal to the line h_{-n+1} the perpendicular meets the line h_{-n+1} in a point A_{-n+1}. Now drop the perpendicular from the point A_{-n+1} to the line h_2 and obtain the point A_{-n+2} as intersection point with the line h_{-n+2}. Continue this procedure as long as you have to drop the perpendicular from the point A_n to the line h_0. The theorem yields that the figure closes i.e. that this perpendicular meets the line h_{-n} in the chosen point A_{-n}.

Example. We show two examples of 7-gons, i.e. $n = 3$. There is no requirement on the polygon $\mathcal{P}$ of convexity or being non-crossed (Figure 2).

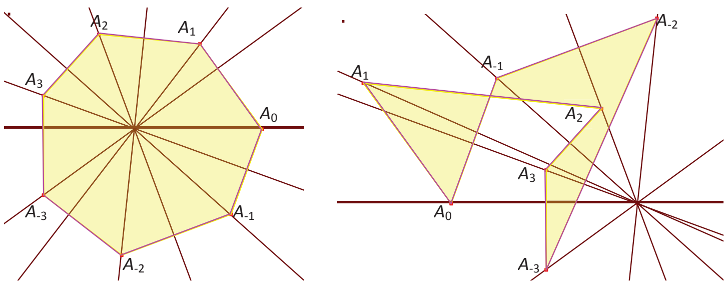


FIGURE 2. Orthocenters of 7-gons

2. Generalizations of Pappus' Theorem

The content of this section was also discussed with Pickert during the last year of his life. Let $\mathcal{P}$ be a plane polygon with $2n$ (mutually different) vertices, $n > 1$. The vertices A_i are indexed by $1, 2, \ldots, 2n$ considered as elements of the factor group $\mathbb{Z}_{2n} = \mathbb{Z}/(2n)\mathbb{Z}$.

Definition 2.1. The pair $(A_iA_{i+1}, A_{i+n}A_{i+n+1})$, $i = 1, 2, \ldots, 2n$, of sides is called *pair of opposite sides.*

So, there are n pairs of opposite sides for each such polygon. Recall now the Pappus' original theorem which plays an essential role in the foundations of geometry.

Theorem 2.2. *Let be given a plane hexagon $A_1A_2 \dots A_6$ such that the vertices with even indices are collinear and also the vertices with odd index (but not all vertices on a line). If two pairs of opposite sides consist of parallel lines so the third pair of opposite sides consists also of parallel lines (Figure 3).*

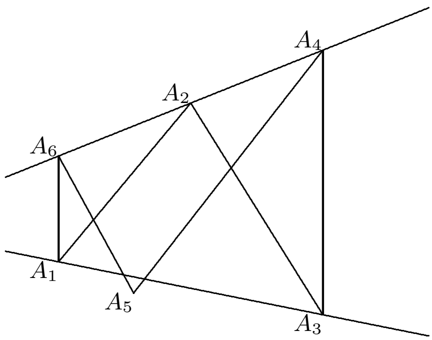


FIGURE 3. $(A_1A_2 \parallel A_4A_5) \wedge (A_2A_3 \parallel A_5A_6) \Longrightarrow (A_3A_4 \parallel A_6A_1)$

Proofs of this theorem for the real plane are well known. It is also the special case of $n = 2$ in the following generalization which will be proved.

Theorem 2.3. *Let be given a polygon $A_1A_2 \dots A_{2n+2}$, n even in the real plane, such that the vertices with even indices are collinear and also the vertices with odd index (but not all vertices on a line). If n pairs of opposite sides consist of parallel lines so the remaining pair of opposite sides consists also of parallel lines.*

Proof. The key to our proof[2] is the fact that in the real plane the group of dilations with a fixed center – proper or improper – is commutative.

Let g denote the supporting line of the vertices with odd indices, h the supporting line of the vertices with even index, and P the proper or improper intersection point of these lines which are different by hypothesis.

It is assumed that the pairs $(A_iA_{i+1}, A_{i+n+1}A_{i+n+2})$ consist of parallel lines, for $i = 1, 2, \dots n$, without loss of generality. Then it is to show that also pair $(A_{n+1}A_{n+2}, A_{2n+2}A_1)$ consists of parallel lines.

Now dilatations δ_I with center P are defined, for $i = 1, 2, \dots n$, by

$$\delta_i(A_i) = A_{i+n+2} \quad \text{if } i \text{ odd},$$
$$\delta_i(A_{i+n+1}) = A_{i+1} \quad \text{if } i \text{ even}.$$

[2]The referee wrote about this theorem: "I have seen it MANY times before. (Sorry, but I have no handy reference.) In fact, Theorem 2.3 is just the original Theorem of Pappus applied multiple times. It has an easy proof by induction (as does Theorem 2.6 of Mbius)". We present this proof - as the referee sketchded it - as proof for Theorem 2.6

If the point P is improper one has translations. Since dilatations map lines onto parallel lines the assumed parallelisms imply

$$\begin{aligned} \delta_i(A_{i+1}) &= A_{i+n+1} \quad \text{if } i \text{ odd},\\ \delta_i(A_{i+n+2}) &= A_i \qquad \text{if } i \text{ even}. \end{aligned}$$

The composition of two subsequent of these dilations yields

$$\begin{aligned} \delta_{i+1}\delta_i(A_i) = \delta_{i+1}(A_{i+n+2}) &= A_{i+2} \quad \text{if } i \text{ odd},\\ \delta_{i-1}\delta_i(A_{i+n+2}) = \delta_{i-1}(A_i) &= A_{i+n} \quad \text{if } i \text{ even}. \end{aligned}$$

Since n is even one obtains

$$\begin{aligned} \delta_n\delta_{n-1}\ldots\delta_1(A_1) &= A_{n+1},\\ \delta_1\delta_2\ldots\delta_n(A_{2n+2}) &= A_{n+2}. \end{aligned}$$

As mentioned above the group of dilatations is commutative one has

$$\delta_n\delta_{n-1}\ldots\delta_1 = \delta_1\delta_2\ldots\delta_n.$$

Let δ denote this composite dilatation. The given formulas show that δ maps the line A_1A_{2n+2} onto the line $A_{n+1}A_{n+2}$. So these lines are parallel as claimed. □

Remark 2.4. Pickert asked why n in this theorem is assumed to be even. The reason is that the configuration does not "close" for odd n. To see this for odd $n > 1$ note that the configuration is determined by $n+2$ cconsecutive vertices, A_1, A_2, ..., A_{n+2} say. Let – as in the preceding proof – g denote the line containing the given vertices with odd index and h the line containing the given vertices with even index. The requirement $A_1A_2 \parallel A_{n+2}A_{n+3}$ forces A_{n+3} to be the intersection point of the line h and the line parallel to A_1A_2 passing through the point A_{n+2} – recall n odd implies $n+3$ even. In the same way the vertices A_{n+4}, ..., A_{2n+2} are determined. Having obtained the vertex A_{2n+2} the configuration should close in the sense that the line parallel to $A_{n+1}A_{n+2}$ through the point A_{2n+2} passes through the point A_1. This is never true when n is odd and greater than 1. As in the accompanying diagram (Figure 4) showing $n = 3$, for example, define A_1' to be the intersection of g with the line through A_8 that is parallel to A_4A_5; then the 5-gon $A_1A_2\ldots A_5$ is the image of $A_5A_6\ldots A_1'$ under a dilatation centered at P the intersection point P of the lines g and h. Consequently A_1' could never coincide with A_1 (unless, of course, the dilatation is the identity and the figure collapses).
In case $n = 1$ we have a quadrangle. There it is possible that both pairs of opposite sides consist of parallel lines. Then we have a parallelogram the diagonals of which serve as the lines g and h.

The 16-years-old Blaise Pascal discovered another generalization of Pappus' theorem. Recall that a pair of lines (g, h) can be considered as a degenerated conic. The theorem is given in a stronger projective version ([3], pp. 74-76).

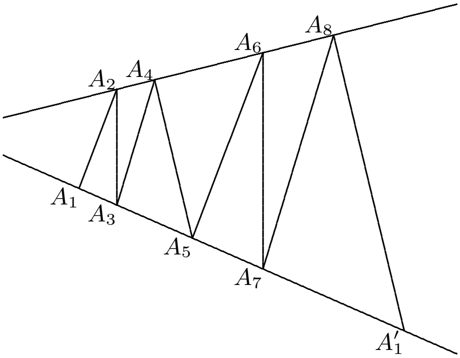


FIGURE 4. Generalized Pappus configuration for $n = 3$

Theorem 2.5. *The intersection points of the three pairs of opposite sides of a hexagon are collinear if and only if the hexagon is inscribed a conic.* (Figure 5)

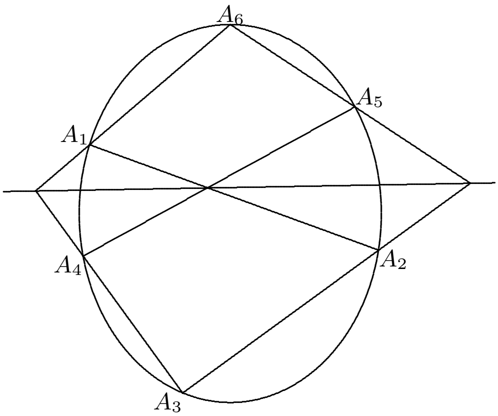


FIGURE 5. Pascal's Theorem

The author conjectured and discussed with Pickert the following generalization of the if-part of Theorem 2.5. While this manuscript was being prepared he found in the internet that it has already been stated by August Ferdinand Möbius ([8]).

Theorem 2.6. *Let a polygon with $2n+2$ vertices, n even, be inscribed a conic. If the intersection points of n pairs of opposite sides belong to one line then the intersection point of the remaining pair of opposite sides lies also on this line* (Figure6)*.*

The proof given by August Ferdinand Möbius covers the case where a projective transformation is possible such that the conic is mapped on a circle and

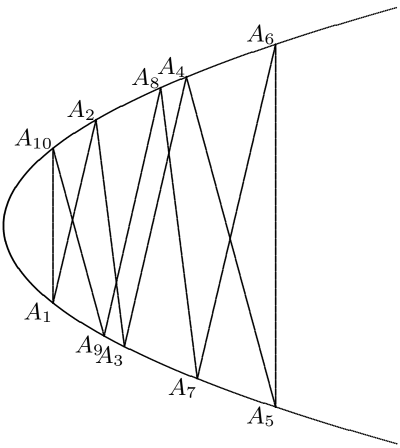


FIGURE 6. Generalized Pascal's Theorem for $n = 4$

the supporting line of the intersection points on the line at infinity. He treats the case when the conic is nondegenerate and the supporting line meets the conic in one or two points only with the remark "Daß es aber auch dann gilt, wenn l den Kegelschnitt berührt, oder schneidet, kann, wenn auch nicht auf diesem einfachen Wege der Projection, doch durch Hinzufügung einer einfachen analytischen Betrachtung gezeigt werden" (That it [the theorem] also holds if l touches or cuts the conic can be shown, although not in this simple manner of projection, but by addition of a simple analytic consideration). Unfortunately the present author does not see such a simple analytic proof. The diagram above shows a decagon inscribed a parabola.

Proof. This proof – communicated by the referee – works by induction on the even numbers. Starting with $n = 2$ one has Pascal's Theorem. The induction step proceeds from $2n+2$ to $2n+6$. Let be given a $(2n+6)$-gon $A_1A_2 \dots A_{2n+6}$ inscribed in a conic such that $A_kA_{k+1} \parallel A_{k+n+3}A_{k+n+4}$ for $k = 1, 2, \dots n+2$. We have to show $A_{n+3}A_{n+4} \parallel A_{2n+6}A_1$.

The induction hypothesis is applied three times: first to the $(2n + 2)$-gon $A_1A_2 \dots A_{n+1}A_{n+4}A_{n+5} \dots A_{2n+4}$ giving $A_{n+1}A_{n+4} \parallel A_{2n+4}A_1$, second to the $(2n + 2)$-gon $A_3A_4 \dots A_{n+3}A_{n+6}A_{n+7} \dots A_{2n+6}$ giving $A_{n+3}A_{n+6} \parallel A_{2n+6}A_3$, third to the $(2n+2)$-gon $A_1A_{2n+4}A_{2n+3} \dots A_{n+6}A_{n+3}A_{n+4}A_{n+1} \dots A_3A_{2n+6}$ giving the desired result. □

3. The Projective Butterfly Theorem

In the year 1814 Reverend T. Scurr from Hexham[3], a small town in the north of England, posed the following question in *The Gentleman's Diary*[11] which

[3]We don't know anything more about his biography.

was answered in the next issue of the journal by several people [6]. Only one of them is nowadays known, William G. Horner (1786–1837), the namesake of the "Horner scheme" for evaluating polynomials [10].

Let AB any chord in a circle, through I the middle point of which let two other chords be drawn, the former MIN, meeting the circle in M, N, the latt OIL, meeting the circle in O, L; join the points L, M, and N, O, by right lines which cut the chord AB in P and Q respectively: then is AP = PQ. Required a demonstration.

We state the question as theorem in a form similar to that given by Coxeter and Greitzer [3, pp. 45-46] using the notation there.

Theorem 3.1 (Butterfly Theorem). *Through the midpoint M of a chord $[PQ]$ of a circle $\mathcal{K}$. any other chords $[AB]$ and $[CD]$ are drawn with A and C on the same side of the line PQ[4]; chords $[AD]$ and $[BC]$ meet $[PQ]$ at points X and Y. Then M is the midpoint of the segment $[XY]$* (Figure 7).

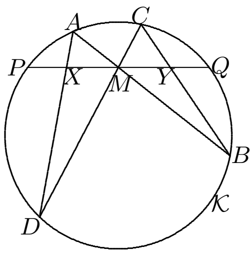


FIGURE 7. The original Butterfly Theorem

The shape of the diagram should explain the denomination "Butterfly Theorem".

By the end of the 19th century Mackay[5] published a generalization of the Butterfly Theorem.

Theorem 3.2. *Given a complete cyclic quadrangle; if any line not parallel to a side of the quadrangle[6] cuts two opposite sides in two different points with equal distances from the center of the circumcircle, it cuts every pair of opposite sides at equal distances from the center of the circumcircle* (Figure 8).

The Butterfly Theorem gained interest at the end of the 20th and at the start of the 21th century, see e.g. the illuminating paper by Leon Bankoff [1].

[4]The assumption "A and C on the same side of the line PQ" is not explicitly formulated in the statement of the Butterfly Theorem in [3], but implicitly used in the proof. The case "A and C on different sides of the line PQ" is given there as a problem for the reader.

[5]John Sturgeon Mackay (1843-1914) was the first president of the Edinburgh Mathematical Society which – in contrast to other learned societies – was founded mainly by school teachers [9].

[6]This assumption is not really necessary. One can allow the equal distances being infinite.

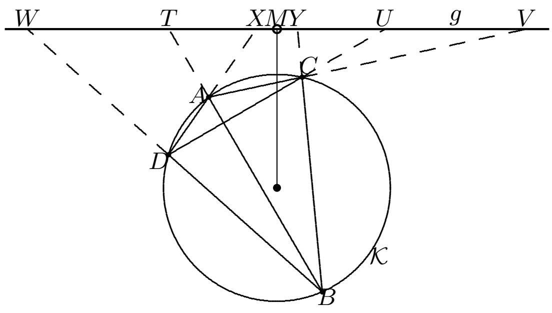


FIGURE 8. Mackay's Theorem

Vladimir Volenec extended Mackay's theorem from circles to conics [13] in an real affine plane.

Theorem 3.3. *Given a complete quadrangle inscribed in a conic; if a line g not parallel to a side of the quadrangle but conjugate to a diameter of the conic cuts two opposite sides in two points with equal distances from the point of intersection M of the line g with the named diameter, it cuts every pair of opposite sides at equal distances from the point M* (Figure 9)*.*

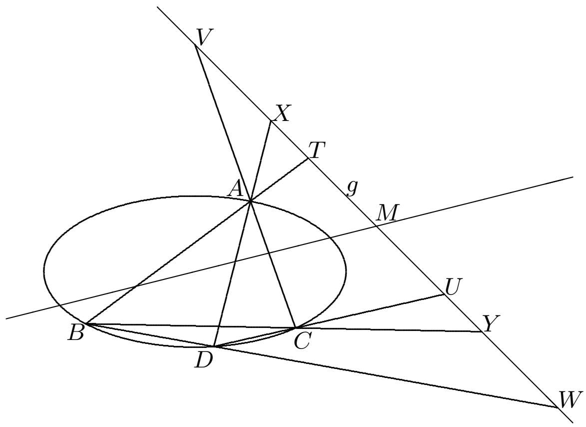


FIGURE 9. Volenec's Theorem

This suggests a projective version from which the theorems 3.1, 3.2, 3.3 can be easily derived. For the following some notation used in this diagram is fixed - the playground is a fixed Pappian projective plane in which the diagonal points of a quadrangle are not collinear: Given a quadrangle $ABCD$ and a line g – which is always assumed not to pass through any of the vertices – one has the quadrangular set $\{T, U, V, W, X, Y\}$ with $T = g(AB)$, $U = g(CD)$, $V = g(AC)$, $W = g(BD)$, $X = g(AD)$, and $Y = g(BC)$.

Theorem 3.4. *Let the complete quadrangle $ABCD$ be inscribed a conic $\mathcal{K}$, g a line not through any vertex of the quadrangle and not a tangent the conic, L a point on the line g, k the polar of L with respect to the conic $\mathcal{K}$, $M = gk$. If one of the quadruples (T, U, L, M), (V, W, L, M), (X, Y, L, M) forms a harmonic set, so do the other two* (Figure 10).

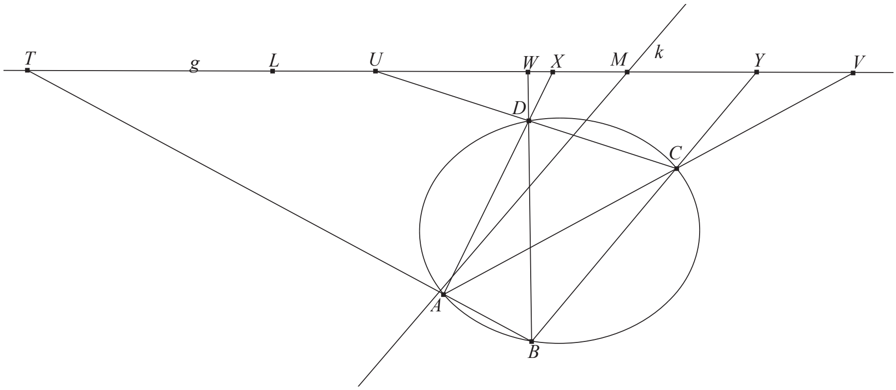


FIGURE 10. Projective Butterfly Theorem

Proof. Without loss of generality one may assume that the quadruple $(TULM)$ forms a harmonic set. Then it is to prove that the quadruples $(VWLM)$ and $(XYLM)$ are also harmonic sets.

The involution J on the line g defined by the quadrangle [2, Theorem 5.33] interchanges the point T with the point U, the point V with the point W, and the point X with the point Y. By Desargues' Involution Theorem [12, Chapter V, Theorem 19] the conic $\mathcal{K}$ also meets the line g in two points P and Q, that are also exchanged by the involution J. Since the points L and M are conjugate with respect to the conic the quadruple $(PQLM)$ forms a harmonic set. Since the quadruple $(TULM)$ the unique involution with fixed points L and M [2, Theorem 5.32], stating that an involution is determined by two of its conjugate pairs, must interchange P with Q and T with U. Thus again by [2, Theorem 5.32] this involution is the involution J. Because the points L and M are the fixed points of the involution J the conclusion follows from [2, Theorem 5.41]: Any involution that has an invariant point L has another invariant point M which is the harmonic conjugate of L with respect to any pair of distinct corresponding points.[7] □

It is interesting to look a the cases which arise from the infinite distances in the affine version.

[7]This proof which replaces the original proof of the author was provided by Chris Fisher based on Bankoff's paper [1],

Corollary 3.5. *Let the complete quadrangle $ABCD$ be inscribed a conic $\mathcal{K}$, g a line not through any vertex of the quadrangle, L a point on the line g, k the polar of L with respect to the conic $\mathcal{K}$, $M = gk$. If $T = U = L$ then the quadruples (V, W, L, M), (X, Y, L, M) form harmonic sets* (Figure 11).

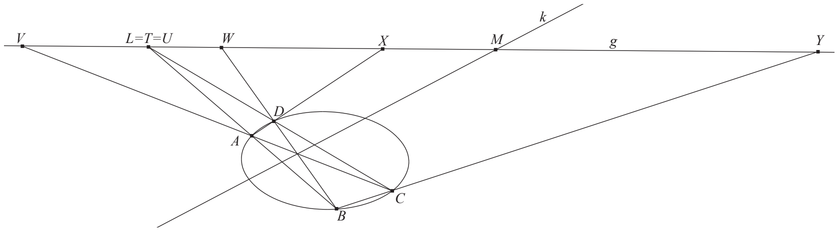


FIGURE 11. A special Projective Butterfly Theorem

Corollary 3.6. *Let the complete quadrangle $ABCD$ be inscribed a conic $\mathcal{K}$, g a line not through any vertex of the quadrangle, L a point on the line g, k the polar of L with respect to the conic $\mathcal{K}$, $M = gk$. If $T = L$ and the quadruple (X, Y, L, M) forms a harmonic sets then $U = T$ and the quadruple (V, W, L, M) also forms a harmonic set.*

But a degenerate case has been overlooked in the proof of Theorem 3.3 in [13]. If the conic is a hyperbola and the line g an asymptote then the line g itself is the diameter appearing in the statement of the theorem and every point on the line g may serve as point M. It is worthwhile to state this special case separately.

Theorem 3.7. *Given a complete quadrangle inscribed in a hyperbola; the three pairs of opposite sides cut an asymptote in segments with the same midpoint* (Figure 12).

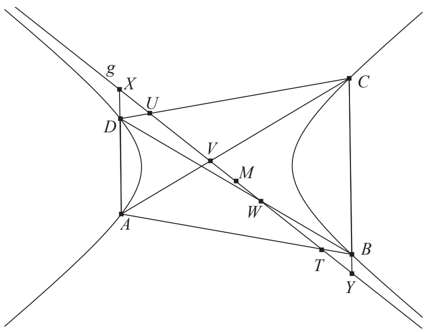
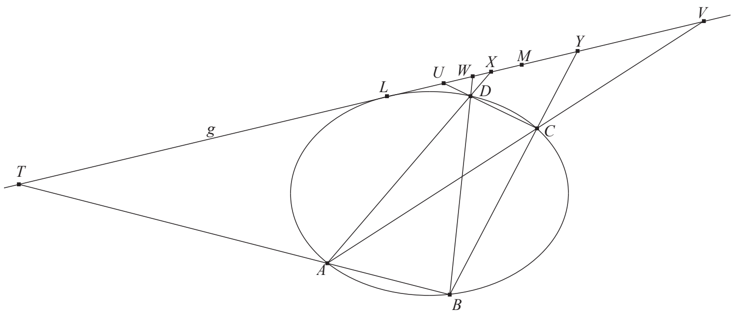


FIGURE 12. Hyperbola and Asymptote

This theorem has the following projective version from which the affine case follows easily.

Theorem 3.8. *Let the complete quadrangle $ABCD$ be inscribed a conic $\mathcal{K}$, g a tangent to the conic $\mathcal{K}$ not through any vertex of the quadrangle, L the point of tangency. Then the triples (T, U, L), (V, W, L), and (X, Y, L) have the same harmonic conjugate of L.*



Proof. This is just another version of the Corollary to Desarues' Involution Theorem stated in [12, Section 46] □

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