13. Proof of the Prime Number Theorem

13.1. In this chapter we will prove the prime number theorem

$$\pi(x) \sim \frac{x}{\log x} \quad \text{for } x \to \infty.$$

As we have seen in corollary 11.11, this is equivalent to the asymptotic relation

$$\psi(x) \sim x \quad \text{for } x \to \infty.$$

To prove this, we use the Mellin transform of ψ, calculated in theorem 12.4

$$\int_1^\infty \psi(x)\, x^{-s}\, \frac{dx}{x} = -\frac{\zeta'(s)}{s\zeta(s)} \quad \text{for } \mathrm{Re}(s) > 1.$$

A first step is

13.2. Proposition. *The following improper integral exists:*

$$\int_1^\infty \Big(\frac{\psi(x)}{x} - 1\Big)\, \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \Big(\frac{\psi(x)}{x} - 1\Big)\, \frac{dx}{x}.$$

Proof. We write the Mellin transform of ψ as a Laplace transform

$$-\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty \psi(e^x)e^{-sx}dx = \int_0^\infty \frac{\psi(e^x)}{e^x}e^{-(s-1)x}dx$$

Since

$$\int_0^\infty e^{-(s-1)x}dx = \frac{1}{s-1} \quad \text{for } \mathrm{Re}(s) > 1,$$

we get for $\mathrm{Re}(s) > 1$

$$\int_0^\infty \Big(\frac{\psi(e^x)}{e^x} - 1\Big)e^{-(s-1)x}dx = -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} =: F(s).$$

The zeta function has a pole of order 1 at $s = 1$, hence $\zeta'(s)/(s\zeta(s))$ has a pole of order 1 with residue -1 at $s = 1$. It follows that F is holomorphic at $s = 1$. We now use the fact that the zeta function has no zeroes on the line $\mathrm{Re}(s) = 1$ and get that the function F can be continued holomorphically to some neighborhood of the closed halfplane $\mathrm{Re}(s) \geq 1$. The Tauberian theorem 12.5 of Ingham/Newman can be applied to the above Laplace transform (after a coordinate change $\tilde{s} = s - 1$), yielding the existence of the improper integral

$$\int_0^\infty \Big(\frac{\psi(e^x)}{e^x} - 1\Big)\, dx.$$

By the substitution $\tilde{x} = e^x$ this is nothing else than the improper integral

$$\int_1^\infty \Big(\frac{\psi(x)}{x} - 1\Big)\,\frac{dx}{x},$$

which proves the proposition.

13.3. Lemma. *Let $g : [1,\infty[\, \to \mathbb{R}$ be a monotonically increasing function such that the improper integral*

$$\int_1^\infty \Big(\frac{g(x)}{x} - 1\Big)\,\frac{dx}{x}$$

exists. Then

$$\lim_{x\to\infty} \frac{g(x)}{x} = 1.$$

Remark. In general, the existence of an improper integral $\int_1^\infty f(x)\frac{dx}{x}$ does not imply $\lim\limits_{x\to\infty} f(x) = 0$, as can be seen by the example

$$\int_1^\infty \sin x\,\frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{\sin x}{x}\,dx.$$

That this improper integral converges follows from the Leibniz criterion for the convergence of alternating series.

Proof. $\lim\limits_{x\to\infty} g(x)/x = 1$ is equivalent to the following two assertions

(1) $\quad \limsup\limits_{x\to\infty} \dfrac{g(x)}{x} \le 1,$

(2) $\quad \liminf\limits_{x\to\infty} \dfrac{g(x)}{x} \ge 1.$

Proof of (1). If this is not true, there exists an $\varepsilon > 0$ and a sequence (x_ν) with $x_\nu \to \infty$ such that

$$g(x_\nu) \ge (1+\varepsilon)x_\nu \quad \text{for all } \nu.$$

Since g is monotonically increasing, it follows that

$$\int_{x_\nu}^{(1+\varepsilon)x_\nu} \Big(\frac{g(x)}{x} - 1\Big)\,\frac{dx}{x} \ge \int_{x_\nu}^{(1+\varepsilon)x_\nu} \Big(\frac{(1+\varepsilon)x_\nu}{x} - 1\Big)\,\frac{dx}{x} = [\text{Subst. } t = \tfrac{x}{x_\nu}]$$

$$= \int_1^{1+\varepsilon} \Big(\frac{1+\varepsilon}{t} - 1\Big)\,\frac{dt}{t} = \alpha(\varepsilon) > 0,$$

where $\alpha(\varepsilon)$ is a positive constant independent of ν (the function $\frac{1+\varepsilon}{t} - 1$ is continuous and positive on the interval $[1, 1+\varepsilon[$). But this contradicts the Cauchy criterion for the existence of the improper integral $\int_1^\infty \left(\frac{g(x)}{x} - 1\right) \frac{dx}{x}$.

Remark. The Cauchy criterion for the existence of the improper integral $\int_a^\infty f(x)dx$ can be formulated as follows: For every $\varepsilon > 0$ there exists an $R_0 \geq a$ such that

$$\Big|\int_R^{R'} f(x)dx\Big| < \varepsilon \quad \text{for all } R, R' \text{ with } R' \geq R \geq R_0.$$

Proof of (2). If this is not true, there exists an $\varepsilon > 0$ and a sequence (x_ν) with $x_\nu \to \infty$ such that

$$g(x_\nu) \leq (1-\varepsilon)x_\nu \quad \text{for all } \nu.$$

Since g is monotonically increasing, it follows that

$$\int_{(1-\varepsilon)x_\nu}^{x_\nu} \Big(\frac{g(x)}{x} - 1\Big)\frac{dx}{x} \leq \int_{(1-\varepsilon)x_\nu}^{x_\nu} \Big(\frac{(1-\varepsilon)x_\nu}{x} - 1\Big)\frac{dx}{x} = [\text{Subst. } t = \tfrac{x}{x_\nu}]$$

$$= \int_{1-\varepsilon}^{1} \Big(\frac{1-\varepsilon}{t} - 1\Big)\frac{dt}{t} = -\beta(\varepsilon) < 0,$$

where $\beta(\varepsilon)$ is a positive constant independent of ν (the function $\frac{1-\varepsilon}{t} - 1$ is continuous and negative on $]1-\varepsilon, 1]$). This contradicts the Cauchy criterion for the existence of the improper integral $\int_1^\infty \left(\frac{g(x)}{x} - 1\right) \frac{dx}{x}$. Therefore (2) must be true, which completes the proof of the lemma.

13.4. Theorem (Prime number theorem). *The prime number function*

$$\pi(x) := \#\{p \in \mathbb{N}_1 : p \text{ prime and } p \leq x\}$$

satisfies the asymptotic relation

$$\pi(x) \sim \frac{x}{\log x} \quad \text{for } x \to \infty.$$

Proof. Lemma 13.3 applied to proposition 13.2 yields $\psi(x) \sim x$, which is by corollary 11.11 equivalent to $\pi(x) \sim x/\log x$, q.e.d.

The following corollary is a generalization of Bertrand's postulate (theorem 11.13).

13.5. Corollary. *For every $\varepsilon > 0$ there exists an $x_0 \geq 1$ such that for all $x \geq x_0$ there is at least one prime p with*

$$x < p \leq (1+\varepsilon)x.$$

Proof. By the prime number theorem

$$\lim_{x\to\infty}\frac{\pi((1+\varepsilon)x)}{\pi(x)} = \lim_{x\to\infty}\frac{(1+\varepsilon)x}{\log(1+\varepsilon)+\log x}\cdot\frac{\log x}{x} = 1+\varepsilon.$$

Therefore there exists an x_0 such that $\pi((1+\varepsilon)x) > \pi(x)$ for all $x \geq x_0$, hence there must be a prime p with $x < p \leq (1+\varepsilon)x$, q.e.d.

13.6. Corollary. *Let p_n denote the n-th prime (in the natural order by size). Then we have the asymptotic relation*

$$p_n \sim n\log n \quad \text{for } n\to\infty.$$

Proof. By the prime number theorem, we have the following asymptotic relation for $n\to\infty$

$$\pi(n\log n) \sim \frac{n\log n}{\log(n\log n)} = \frac{n\log n}{\log n+\log\log n} = \frac{n}{1+\frac{\log\log n}{\log n}} \sim n.$$

Since $\pi(p_n) = n$ by definition, the assertion follows immediately from the next lemma.

13.7. Lemma. *Let $f, g : \mathbb{N}_1 \to \mathbb{R}_+$ be two functions with $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ and*

$$\pi(f(n)) \sim \pi(g(n)) \quad \text{for } n\to\infty.$$

Then we have also

$$f(n) \sim g(n) \quad \text{for } n\to\infty.$$

Proof. We have to show

$$(1) \quad \limsup_{n\to\infty}\frac{f(n)}{g(n)} \leq 1 \qquad \text{and} \qquad (2) \quad \limsup_{n\to\infty}\frac{g(n)}{f(n)} \leq 1.$$

To prove (1), assume this is false. Then there exists an $\varepsilon > 0$ and a sequence (n_ν) with $n_\nu \to \infty$ such that

$$f(n_\nu) \geq (1+\varepsilon)g(n_\nu) \quad \text{for all } \nu.$$

Since

$$\lim_{\nu\to\infty}\frac{\pi((1+\varepsilon)g(n_\nu))}{\pi(g(n_\nu))} = 1+\varepsilon,$$

cf. the proof of corollary 13.5, this implies

$$\limsup_{\nu\to\infty}\frac{\pi(f(n_\nu))}{\pi(g(n_\nu)} \geq 1+\varepsilon,$$

contradicting the hypothesis $\pi(f(n)) \sim \pi(g(n))$. Therefore (1) must be true. Assertion (2) follows from (1) by interchanging the roles of f and g.