12. Laplace and Mellin Transform

12.1. Laplace Transform. Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a measurable function such that $|f(x)|e^{-\sigma_0 x}$ is bounded on $\mathbb{R}_+$ for some $\sigma_0 \in \mathbb{R}$. Then the integral

$$F(s) = \int_0^\infty f(x)e^{-sx}dx$$

exists for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$ and represents a holomorphic function in the halfplane

$$H(\sigma_0) = \{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_0\}$$

F is called the Laplace transform of f.

Remark. Measurable here means Lebesgue measurable. In our applications, f will always be at least piecewise continuous. Hence the reader who does not feel confortable with Lebesgue integration theory may assume f piecewise continuous.

The existence of the integral follows from the estimate

$$|f(x)e^{sx}| \le Ke^{-(\sigma-\sigma_0)x}, \quad \sigma := \mathrm{Re}(s) > \sigma_0,$$

where K is an upper bound for $|f(x)|e^{\sigma_0 x}$ on $\mathbb{R}$.

Example. Let $f(x) = 1$ for all $x \in \mathbb{R}_+$. The Laplace transform of this function is

$$F(s) = \int_0^\infty e^{-sx}dx = \lim_{R\to\infty}\Big[-\frac{e^{-sx}}{s}\Big]_{x=0}^{x=R} = \lim_{R\to\infty}\frac{1-e^{-sR}}{s} = \frac{1}{s} \quad \text{for } \mathrm{Re}(s) > 0.$$

12.2. *Relation between Laplace and Fourier transform.*
We set $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. Then the formula for the Laplace transform becomes

$$F(\sigma+it) = \int_0^\infty f(x)e^{-\sigma x}e^{-itx}dx = \int_{-\infty}^\infty g(x)e^{-itx}dx,$$

where

$$g(x) = \begin{cases} f(x)e^{-\sigma x} & \text{for } x \ge 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Therefore the function $t \mapsto F(\sigma+it)$ can be regarded (up to a normalization constant) as the Fourier transform of the function g.

12.3. Mellin Transform. The Mellin transform is obtained from the Laplace transform by a change of variables. With the substitution

$$x = \log t, \quad dx = \frac{dt}{t},$$

the formula for the Laplace transform becomes

$$F(s) = \int_1^\infty f(\log t)\, t^{-s}\, \frac{dt}{t}.$$

This can be viewed as a transformation of the function $g(t) := f(\log t)$, $t \geq 1$, and leads to the following definition.

Definition. Let $g : [1, \infty[\to \mathbb{R}$ a measurable function such that $g(x)x^{-\sigma_0}$ is bounded on $[1, \infty[$ for some $\sigma_0 \in \mathbb{R}$. Then the integral

$$G(s) = \int_1^\infty g(x)x^{-s}\, \frac{dx}{x}$$

exists for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$. The function G is holomorphic in the halfplane $H(\sigma_0)$ and is called the Mellin transform of g.

Remark. There exists a generalization of the Mellin transform where the integral is extended from 0 to ∞. An example is the Euler integral for the Gamma function

$$\Gamma(s) = \int_0^\infty e^{-x}x^{-s}\, \frac{dx}{x}.$$

This generalized Mellin transform corresponds to the "two-sided" Laplace transform

$$F(s) = \int_{-\infty}^\infty f(x)e^{-sx}dx$$

12.4. Theorem. *The Mellin transform of the Chebyshev ψ-function is*

$$\int_1^\infty \psi(x)\, x^{-s}\, \frac{dx}{x} = -\frac{\zeta'(s)}{s\zeta(s)} \quad \text{for } \mathrm{Re}(s) > 1.$$

Proof. It follows from theorems 11.3 and 11.10 that $\psi(x)/x$ is bounded, hence the Mellin transform of ψ exists for $\mathrm{Re}(s) > 1$. We apply the Abel summation theorem 11.4 to the sum $\sum_{n \leqslant x} \frac{\Lambda(n)}{n^s}$. Since

$$\frac{d}{dx}\frac{1}{x^s} = -s\, \frac{1}{x^{s+1}},$$

we obtain

$$\sum_{n \leqslant x} \frac{\Lambda(n)}{n^s} = \frac{\psi(x)}{x^s} + s\int_1^x \frac{\psi(t)}{t^{s+1}}\, dt.$$

Letting $x \to \infty$, we get $\psi(x)/x^s \to 0$ for $\mathrm{Re}(s) > 1$, and using theorem 11.8

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = s \int_1^\infty \frac{\psi(t)}{t^{s+1}}\, dt, \quad \text{q.e.d.}$$

12.5. Theorem (Tauberian theorem of Ingham and Newman). *Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a measurable bounded function and*

$$F(s) = \int_0^\infty f(x) e^{-sx} dx, \quad \mathrm{Re}(s) > 0,$$

its Laplace transform. Suppose that F, which is holomorphic in

$$H(0) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\},$$

admits a holomorphic continuation to some open neighborhood U of $\overline{H(0)}$. Then the improper integral

$$\int_0^\infty f(x) dx = \lim_{R \to \infty} \int_0^R f(x) dx$$

exists and one has

$$F(0) = \int_0^\infty f(x) dx,$$

where $F(0)$ denotes the value at 0 of the continued function.

Proof. For a real parameter $R > 0$ define the function

$$F_R(s) := \int_0^R f(x) e^{-sx} dx.$$

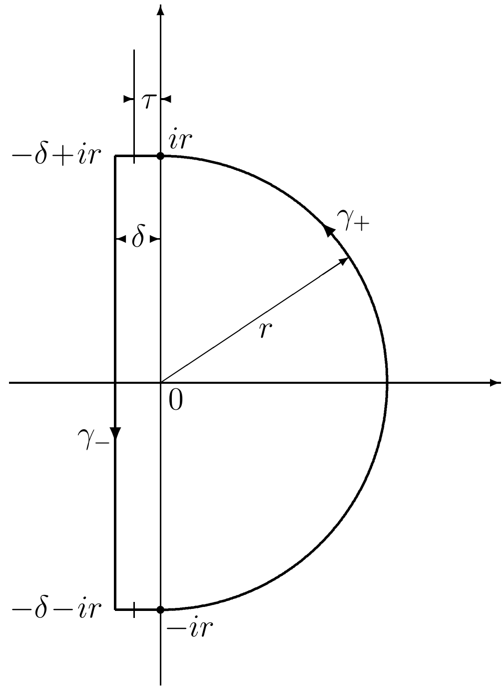
Since the integration interval $[0, R]$ is compact, F_R is holomorphic in the whole plane $\mathbb{C}$. The assertion of the theorem is equivalent to

$$\lim_{R \to \infty} (F(0) - F_R(0)) = 0.$$

The function $F - F_R$ is holomorphic in $U \supset \overline{H(0)}$, therefore its value at the point 0 can be calculated by the Cauchy formula.

$$F(0) - F_R(0) = \frac{1}{2\pi i} \int_\gamma (F(s) - F_R(s)) \frac{1}{s}\, ds.$$

Here the curve $\gamma = \gamma_+ + \gamma_-$ is chosen as indicated in the following figure. γ_+ is a semi-circle of radius $r > 0$ with center 0 in the right halfplane from $-ir$ to ir, and γ_- consists of three straight lines from ir to $-\delta + ir$, from $-\delta + ir$ to $-\delta - ir$ and from $-\delta - ir$ to $-ir$. The constant $\delta > 0$ has to be chosen (depending on r) sufficiently small, such that γ and its interior are completely contained in U.



The function $s \mapsto (F(s) - F_R(s))\, e^{Rs}$ is holomorphic in U and for $s = 0$ its value is $F(0) - F_R(0)$. Therefore we have also

$$F(0) - F_R(0) = \frac{1}{2\pi i} \int_\gamma (F(s) - F_R(s))\, e^{Rs}\, \frac{1}{s}\, ds.$$

We still use another trick and write

$$F(0) - F_R(0) = \frac{1}{2\pi i} \int_\gamma (F(s) - F_R(s))\, e^{Rs} \left(\frac{1}{s} + \frac{s}{r^2}\right) ds. \tag{$*$}$$

This is true since the added function

$$s \mapsto (F(s) - F_R(s))\, e^{Rs}\, \frac{s}{r^2}$$

is holomorphic in U, hence its integral over γ vanishes.

Note that for $|s| = r$ one has

$$\left(\frac{1}{s} + \frac{s}{r^2}\right) = \frac{\bar{s}}{s\bar{s}} + \frac{s}{r^2} = \frac{s + \bar{s}}{r^2} = \frac{2\sigma}{r^2}, \quad \text{where } \sigma = \mathrm{Re}(s).$$

For the proof of our theorem, we have to estimate the integral $(*)$.

Let $\varepsilon > 0$ be given. We choose $r := 3/\varepsilon$ and a suitable $\delta > 0$. We estimate the integral in three steps.

1) Estimation of the integral over the curve γ_+.

Since by hypothesis $f : \mathbb{R} \to \mathbb{C}$ is bounded, we may suppose $|f(x)| \le 1$ for all $x \ge 0$. Then for $\sigma = \mathrm{Re}(s) > 0$

$$|F(s) - F_R(s)| = \Big|\int_R^\infty f(x)e^{-sx}dx\Big| \le \int_R^\infty e^{-\sigma x}dx = \frac{e^{-R\sigma}}{\sigma}.$$

With the abbreviation

$$G_1(s) := (F(s) - F_R(s))\, e^{Rs}\left(\frac{1}{s} + \frac{s}{r^2}\right)$$

we get therefore on γ_+

$$|G_1(s)| \le \frac{e^{-R\sigma}}{\sigma}\, e^{R\sigma}\,\frac{2\sigma}{r^2} = \frac{2}{r^2},$$

hence

$$\Big|\frac{1}{2\pi i}\int_{\gamma_+} G_1(s)ds\Big| \le \frac{1}{2\pi}\int_{\gamma_+}\frac{2}{r^2}\,|ds| = \frac{1}{2\pi}\cdot\frac{2}{r^2}\cdot\pi r = \frac{1}{r} = \frac{\varepsilon}{3}.$$

2) Estimation of the integral $\displaystyle\int_{\gamma_-} F_R(s)e^{Rs}\Big(\frac{1}{s} + \frac{s}{r^2}\Big)ds.$

Since F_R is holomorphic in the whole plane, we may replace the integration curve γ_- by a semicircle α of radius r in the halfplane $\mathrm{Re}(s) \le 0$ from ir to $-ir$. For $\sigma = \mathrm{Re}(s) < 0$ we have

$$|F_R(s)| \le \int_0^R e^{-x\sigma}dx = \frac{1}{\sigma}(1 - e^{-R\sigma}) \le \frac{e^{-R\sigma}}{|\sigma|},$$

Therefore the integrand

$$G_2(s) := F_R(s)e^{Rs}\Big(\frac{1}{s} + \frac{s}{r^2}\Big)$$

satisfies the following estimate on the curve α

$$|G_2(s)| \le \big|F_R(s)e^{Rs}\big|\,\frac{2|\sigma|}{r^2} \le \frac{2}{r^2},$$

hence

$$\Big|\frac{1}{2\pi i}\int_\alpha G_2(s)ds\Big| \le \frac{1}{2\pi}\int_\alpha \frac{2}{r^2}|ds| = \frac{1}{\pi r^2}\int_\alpha |ds| = \frac{1}{r} = \frac{\varepsilon}{3}.$$

3) Estimation of the integral $\displaystyle\int_{\gamma_-} F(s)e^{Rs}\Big(\frac{1}{s} + \frac{s}{r^2}\Big)ds.$

The function $s \mapsto F(s)\Big(\frac{1}{s} + \frac{s}{r^2}\Big)$ is holomorphic in a neighborhood of the integration path γ_-. Therefore there exists a constant $K > 0$ such that

$$\Big|F(s)\Big(\frac{1}{s} + \frac{s}{r^2}\Big)\Big| \le K \quad \text{for all } s \text{ on the curve } \gamma_-.$$

Hence the integrand

$$G_3(s) := F(s)e^{Rs}\Big(\frac{1}{s} + \frac{s}{r^2}\Big)$$

satisfies the following estimate on γ_-

$$|G_3(s)| \le Ke^{R\sigma}, \quad \text{where } \sigma = \mathrm{Re}(s).$$

Let τ be some constant with

$$0 < \tau < \delta,$$

whose value will be fixed later. We split the integration curve γ_- into two parts

$$\gamma'_- := \gamma_- \cap \{\mathrm{Re}(s) \ge -\tau\},$$
$$\gamma''_- := \gamma_- \cap \{\mathrm{Re}(s) \le -\tau\}.$$

γ'_- consists of two line segments of length τ each. Let L be the length of γ_-. Then

$$\begin{aligned}\Big|\frac{1}{2\pi i}\int_{\gamma_-} G_3(s)ds\Big| &\le \frac{1}{2\pi}\Big\{\int_{\gamma'_-} Ke^{R\sigma}|ds| + \int_{\gamma''_-} Ke^{R\sigma}|ds|\Big\} \\ &\le \frac{K}{2\pi}\Big\{\int_{\gamma'_-} |ds| + \int_{\gamma''_-} e^{-R\tau}|ds|\Big\} \\ &\le \frac{K}{2\pi}\left(2\tau + Le^{-R\tau}\right).\end{aligned}$$

We now fix a value of $\tau > 0$ such that

$$\frac{K}{2\pi} \cdot 2\tau < \frac{\varepsilon}{6}$$

and choose an $R_0 > 0$ such that

$$\frac{K}{2\pi} \cdot Le^{-R_0\tau} < \frac{\varepsilon}{6}$$

Then we have

$$\Big|\frac{1}{2\pi i}\int_{\gamma_-} G_3(s)ds\Big| < \frac{\varepsilon}{3} \quad \text{for all } R \ge R_0.$$

Putting the estimates of 1), 2) and 3) together we finally get

$$|F(0) - F_R(0)| = \Big|\frac{1}{2\pi i}\int_{\gamma}(F(s) - F_R(s))\, e^{Rs}\left(\frac{1}{s} + \frac{s}{r^2}\right) ds\Big| < \varepsilon$$

for all $R \ge R_0$, q.e.d.