10. Functional Equation of the Zeta Function

10.1. Theorem (Functional equation of the theta function).
The theta series is defined for real $x > 0$ by

$$\theta(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

It satisfies the following functional equation

$$\theta\Big(\frac{1}{x}\Big) = \sqrt{x}\theta(x) \qquad \text{for all } x > 0,$$

i.e.

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/x}.$$

Remarks. a) The theta series, as well as its derivatives, converge uniformly on every interval $[\varepsilon, \infty[$, $\varepsilon > 0$; hence θ is a $\mathcal{C}^\infty$-function on $]0, \infty[$.
b) In the theory of elliptic functions one defines more general theta functions of two complex variables. For $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and $z \in \mathbb{C}$ one sets

$$\vartheta(\tau, z) := \sum_{n\in\mathbb{Z}} e^{i\pi n^2\tau} e^{2\pi i n z}.$$

For fixed τ this is an entire holomorphic function in z, which can be used to construct doubly periodic functions with respect to the lattice $\mathbb{Z} + \mathbb{Z}\tau$. As a function of τ, it is holomorphic in the upper halfplane. The relation to the theta series of theorem 10.1 is

$$\theta(t) = \vartheta(it, 0).$$

Proof. For fixed $x > 0$, we consider the function $F : \mathbb{R} \to \mathbb{R}$,

$$F(t) := \sum_{n\in\mathbb{Z}} e^{-\pi(n-t)^2 x}.$$

The series converges uniformly on $\mathbb{R}$ together with all its derivatives, hence represents a $\mathcal{C}^\infty$-function on $\mathbb{R}$. It is periodic with period 1, i.e. $F(t + 1) = F(t)$ for all $t \in \mathbb{R}$. Therefore we can expand F as a uniformly convergent Fourier series

$$F(t) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n t}$$

where the coefficients c_n are the integrals

$$c_n = \int_0^1 F(t) e^{-2\pi i n t} dt = \sum_{k\in\mathbb{Z}} \int_0^1 e^{-\pi(k-t)^2 x} e^{-2\pi i n t} dt.$$

Now $\int_0^1 e^{-\pi(k-t)^2 x} e^{-2\pi int} dt = \int_k^{k+1} e^{-\pi t^2 x} e^{-2\pi int} dt$ (substitution $\tilde{t} = t - k$), hence

$$c_n = \int_{-\infty}^{\infty} e^{-\pi t^2 x} e^{-2\pi int} dt.$$

For $n = 0$ this is the well known integral of the Gauss bell curve

$$\begin{aligned} c_0 &= \int_{-\infty}^{\infty} e^{-\pi t^2 x} dt = 2\int_0^{\infty} e^{-\pi t^2 x} dt = \frac{2}{\sqrt{\pi x}} \int_0^{\infty} e^{-t^2} dt \\ &= \frac{1}{\sqrt{\pi x}} \int_0^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi x}} \Gamma\Big(\frac{1}{2}\Big) = \frac{1}{\sqrt{x}}. \end{aligned}$$

For general n we write

$$-\pi t^2 x - 2\pi int = -\pi\Big(t\sqrt{x} + \frac{in}{\sqrt{x}}\Big)^2 - \frac{\pi n^2}{x}.$$

This leads to

$$c_n = e^{-\pi n^2/x} \int_{-\infty}^{\infty} e^{-\pi(t\sqrt{x} + in/\sqrt{x})^2} dt.$$

We will prove

$$\int_{-\infty}^{\infty} e^{-\pi(t\sqrt{x} + in/\sqrt{x})^2} dt = \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi t^2} dt = \frac{1}{\sqrt{x}}. \qquad (*)$$

Assuming this for a moment, we get

$$F(t) = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/x} e^{2\pi int}.$$

Setting $t = 0$, it follows

$$F(0) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-\pi n^2/x},$$

which is the assertion of the theorem.

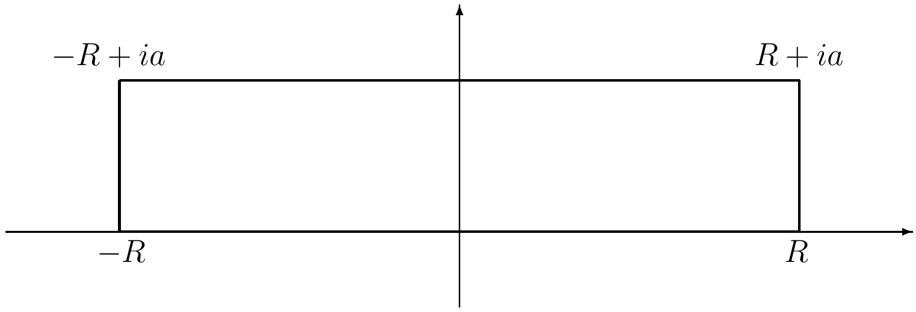
It remains to prove the formula $(*)$. Using the substitution $\tilde{t} = t\sqrt{x}$ we see that

$$\int_{-\infty}^{\infty} e^{-\pi(t\sqrt{x} + in/\sqrt{x})^2} dt = \frac{1}{\sqrt{x}} \int_{-\infty}^{\infty} e^{-\pi(t + in/\sqrt{x})^2} dt$$

With the abbreviation $a := n/\sqrt{x}$ we have to show that

$$\int_{-\infty}^{\infty} e^{-\pi(t+ia)^2} dt = \int_{-\infty}^{\infty} e^{-\pi t^2} dt. \qquad (**)$$

To this end we integrate the holomorphic function $f(z) := e^{-\pi z^2}$ over the boundary of the rectangle with corners $-R, R, R + ia, -R + ia$, where R is a positive real number.



By the residue theorem the whole integral is zero, hence

$$\int_{-R}^{R} f(z)dz = \int_{-R+ia}^{R+ia} f(z)dz - \int_{R}^{R+ia} f(z)dz + \int_{-R}^{-R+ia} f(z)dz$$

Now

$$\int_{-R}^{R} f(z)dz = \int_{-R}^{R} e^{-\pi t^2} dt,$$

$$\int_{-R+ia}^{R+ia} f(z)dz = \int_{-R}^{R} e^{-\pi(t+ia)^2} dt,$$

$$\int_{\pm R}^{\pm R+ia} f(z)dz = i\int_0^a e^{-\pi(R^2-t^2)\mp 2\pi iRt}dt = ie^{-\pi R^2}\int_0^a e^{\pi t^2 \mp 2\pi iRt}dt.$$

We have the estimate

$$\Big|\int_{\pm R}^{\pm R+ia} f(z)dz\Big| \le e^{-\pi R^2}|a|e^{\pi|a|^2},$$

which tends to 0 as $R\to\infty$. This implies

$$\lim_{R\to\infty}\int_{-R}^{R} e^{-\pi t^2}dt = \lim_{R\to\infty}\int_{-R}^{R} e^{-\pi(t+ia)^2}dt,$$

which proves $(**)$ and therefore $(*)$. This completes the proof of the functional equation of the theta function.

10.2. Corollary. *The theta function $\theta(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ defined in the preceding theorem satifies*

$$\theta(x) = O\Big(\frac{1}{\sqrt{x}}\Big) \quad \text{as } x \searrow 0.$$

10.3. Proposition. *For all $s\in\mathbb{C}$ with $\mathrm{Re}(s) > 1$ one has*

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{s/2}\int_0^\infty t^{s/2}\Big(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\Big)\frac{dt}{t}.$$

Remark. The function

$$\psi(t) := \sum_{n=1}^{\infty} e^{-\pi n^2 t}$$

decreases exponentially as $t \to \infty$. One has $\theta(t) = 1 + 2\psi(t)$, hence $\psi(t) = \frac{1}{2}(\theta(t) - 1)$, so corollary 10.2 implies

$$\psi(t) = O\Big(\frac{1}{\sqrt{t}}\Big) \qquad \text{for } t \searrow 0.$$

This shows that the integral exists for $\mathrm{Re}(s) > 1$.

Proof. We start with the Euler integral for $\Gamma(s/2)$,

$$\Gamma\Big(\frac{s}{2}\Big) = \int_0^\infty t^{s/2} e^{-t} \frac{dt}{t},$$

and apply the substitution $\tilde{t} = \pi n^2 t$, where $n \in \mathbb{N}_1$. Since $d\tilde{t}/\tilde{t} = dt/t$, we get

$$\Gamma\Big(\frac{s}{2}\Big) = n^s \pi^{s/2} \int_0^\infty t^{s/2} e^{-\pi n^2 t} \frac{dt}{t}.$$

For $\mathrm{Re}(s) > 1$ we have

$$\begin{aligned} \Gamma\Big(\frac{s}{2}\Big)\zeta(s) &= \sum_{n=1}^{\infty} \Gamma\Big(\frac{s}{2}\Big)\frac{1}{n^s} = \sum_{n=1}^{\infty} \pi^{s/2} \int_0^\infty t^{s/2} e^{-\pi n^2 t} \frac{dt}{t} \\ &= \pi^{s/2} \int_0^\infty t^{s/2} \Big(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\Big) \frac{dt}{t}. \end{aligned}$$

The interchange of summation and integration is allowed by the theorem of majorized convergence for Lebesgue integrals.

10.4. Theorem (Functional equation of the zeta function).
a) *The function*

$$\xi(s) := \pi^{-s/2}\, \Gamma(s/2)\, \zeta(s),$$

which is a meromorphic function in $\mathbb{C}$, *satisfies the functional equation*

$$\xi(1-s) = \xi(s).$$

b) *For the zeta function itself one has*

$$\zeta(1-s) = 2^{1-s} \pi^{-s}\, \Gamma(s) \cos(\tfrac{\pi s}{2})\, \zeta(s).$$

Proof. By the preceding theorem

$$\xi(s) = \int_0^\infty t^{s/2}\psi(t)\frac{dt}{t} \quad \text{with} \quad \psi(t) = \sum_{n=1}^\infty e^{-\pi n^2 t}.$$

The functional equation of the theta function implies for $\psi(t) = \frac{1}{2}(\theta(t) - 1)$

$$\psi(t) = t^{-1/2}\psi(1/t) - \tfrac{1}{2}(1 - t^{-1/2}).$$

We substitute this expression in the integral from 0 to 1:

$$\int_0^1 t^{s/2}\psi(t)\frac{dt}{t} = \int_0^1 t^{(s-1)/2}\psi\Big(\frac{1}{t}\Big)\frac{dt}{t} + \tfrac{1}{2}\int_0^1 (t^{(s-1)/2} - t^{s/2})\frac{dt}{t}.$$

The last integral can be evaluated explicitly (recall that $\mathrm{Re}(s) > 1$):

$$\tfrac{1}{2}\int_0^1 (t^{(s-1)/2} - t^{s/2})\frac{dt}{t} = \frac{1}{s-1} - \frac{1}{s}.$$

For the first integral on the right hand side we use the substitution $\tilde{t} = 1/t$ and obtain

$$\int_0^1 t^{(s-1)/2}\psi\Big(\frac{1}{t}\Big)\frac{dt}{t} = \int_1^\infty t^{(1-s)/2}\psi(t)\frac{dt}{t}.$$

Putting everything together we get

$$\xi(s) = \int_0^\infty t^{s/2}\psi(t)\frac{dt}{t} = \int_1^\infty \big(t^{(1-s)/2} + t^{s/2}\big)\,\psi(t)\frac{dt}{t} + \Big(\frac{1}{s-1} - \frac{1}{s}\Big).$$

The integral on the right hand side converges for all $s \in \mathbb{C}$ to a holomorphic function in $\mathbb{C}$. Thus we have got a representation of the function $\xi(s)$ valid in the whole plane. This representation is invariant under the map $s \mapsto 1 - s$, proving $\xi(1-s) = \xi(s)$, i.e. part a) of the theorem.

To prove part b), we use the equation we just proved:

$$\pi^{-(1-s)/2}\Gamma\Big(\frac{1-s}{2}\Big)\,\zeta(1-s) = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s),$$

yielding

$$\zeta(1-s) = \pi^{1/2-s}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{1-s}{2}\Big)^{-1}\zeta(s).$$

By theorem 9.5.a) we have

$$\Gamma\Big(\frac{1-s}{2}\Big)\Gamma\Big(\frac{1+s}{2}\Big) = \frac{\pi}{\sin(\pi\frac{1+s}{2})} = \frac{\pi}{\cos(\frac{\pi s}{2})},$$

therefore

$$\zeta(1-s) = \pi^{-1/2-s}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{1+s}{2}\Big)\cos(\tfrac{\pi s}{2})\,\zeta(s).$$

Now by theorem 9.5.b)

$$\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{1+s}{2}\Big) = 2^{1-s}\sqrt{\pi}\,\Gamma(s),$$

which implies

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos(\tfrac{\pi s}{2})\,\zeta(s), \qquad \text{q.e.d.}$$

10.5. Corollary. a) *For every integer* $k > 0$

$$\zeta(-2k) = 0.$$

These are the only zeroes of the zeta function in the halfplane $\mathrm{Re}(s) < 0$.

b) $\zeta(0) = -\frac{1}{2}$.

c) *For every integer* $k > 0$

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}.$$

Proof. a) We use the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\Gamma(s)\cos\tfrac{\pi s}{2}\,\zeta(s)$$

$\mathrm{Re}(1-s) < 0$ is equivalent to $\mathrm{Re}(s) > 1$. Since $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$ (theorem 4.5), the only zeroes of the right hand side for $\mathrm{Re}(s) > 1$ come from the cosine function. Now

$$\cos\tfrac{\pi s}{2} = 0 \quad\Longleftrightarrow\quad s = 1 + 2k \quad \text{with } k \in \mathbb{Z}$$

This implies assertion a)

c) From the functional equation we get

$$\zeta(1-2k) = 2^{1-2k}\pi^{-2k}\Gamma(2k)\cos(\pi k)\zeta(2k) = \frac{2}{(2\pi)^{2k}}(2k-1)!\,(-1)^k\zeta(2k).$$

By theorem 5.8.ii)

$$\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

Substituting this in the equation above yields

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}.$$

b) We write the functional equation in the form $\zeta(1-s) = f_1(s)f_2(s)$ with

$$f_1(s) := 2^{1-s}\pi^{-s}\Gamma(s) \quad \text{and} \quad f_2(s) := \cos\tfrac{\pi s}{2}\,\zeta(s).$$

f_1 is holomorphic in a neighborhood of $s = 1$ and $f_1(1) = 1/\pi$. The function f_2 is likewise holomorphic in a neighborhood of $s = 1$, since the pole of the zeta function is cancelled by the zero of the cosine. To calculate $f_2(1)$, we determine the first terms of the Taylor resp. Laurent expansions of the factors.

$$\begin{aligned} \cos\tfrac{\pi s}{2} &= \cos\left(\tfrac{\pi}{2}(s-1) + \tfrac{\pi}{2}\right) = -\sin\left(\tfrac{\pi}{2}(s-1)\right) = -\tfrac{\pi}{2}(s-1) + O((s-1)^3), \\ \zeta(s) &= \frac{1}{s-1} + (\text{holomorphic function}). \end{aligned}$$

Multiplying both expressions yields $f_2(s) = -\frac{\pi}{2} + O(s-1)$, hence $f_2(1) = -\frac{\pi}{2}$. Therefore

$$\zeta(0) = f_1(1)f_2(1) = -\tfrac{1}{2}, \quad \text{q.e.d.}$$

10.6. Theorem. *For all $t \in \mathbb{R}$*

$$\zeta(1+it) \neq 0.$$

Proof. We use the inequality

$$3 + 4\cos t + \cos 2t \geq 0 \qquad \text{for all } t \in \mathbb{R}.$$

This is proved as follows: Since $\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$, we have

$$3 + 4\cos t + \cos 2t = 2(1 + 2\cos t + \cos^2 t) = 2(1+\cos t)^2 \geq 0.$$

Let now $s = \sigma + it$ be a complex number with $\text{Re}(s) = \sigma > 1$. Then

$$\log\zeta(s) = \sum_{p\in\mathbb{P}} \log\frac{1}{1-p^{-s}} = \sum_{p\in\mathbb{P}}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{1}{p^{ks}} = \sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

where

$$a_n = \begin{cases} 1/k, & \text{if } n = p^k \text{ for some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\log|z| = \text{Re}(\log z)$ for every $z \in \mathbb{C}^*$,

$$\log|\zeta(s)| = \sum_{n=1}^{\infty} a_n \text{Re}(n^{-s}) = \sum_{n=1}^{\infty}\frac{a_n}{n^\sigma}\cos(t\log n).$$

Using a trick of v. Mangoldt (1895) we form the expression

$$\log\Big(|\zeta(\sigma)|^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|\Big) = \sum_{n=1}^{\infty}\frac{a_n}{n^\sigma}\Big(\underbrace{3+4\cos(t\log n)+\cos(2t\log n)}_{\geq 0}\Big) \geq 0.$$

Therefore

$$\big|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\big| \geq 1 \quad \text{for all } \sigma > 1 \text{ and } t \in \mathbb{R}.$$

Assume that $\zeta(1+it) = 0$ for some $t \neq 0$. Then the function $s \mapsto \zeta(s)^3\zeta(s+it)^4$ has a zero at $s = 1$, since the pole of order 3 of the function $\zeta(s)^3$ is compensated by the zero of order ≥ 4 of the function $\zeta(s+it)^4$. Therefore

$$\lim_{\sigma \searrow 1}\big|\zeta(\sigma)^3\zeta(\sigma+it)^4\zeta(\sigma+2it)\big| = 0,$$

contradicting the above estimate. Hence the assumption is false, which proves the theorem.

10.7. Riemann Hypothesis. It follows from theorem 10.6 and the functional equation that $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) = 0$. Therefore, besides the trivial zeroes of the zeta function at $s = -2k$, $k \in \mathbb{N}_1$, all other zeroes of the zeta function must satisfy $0 < \mathrm{Re}(s) < 1$. It was conjectured by Riemann in 1859 that all non-trivial zeroes of the zeta function actually have $\mathrm{Re}(s) = \frac{1}{2}$. This is the famous Riemann hypothesis which is still unproven today.