9. The Gamma Function

9.1. Definition. The *Gamma function* is defined for complex z with $\mathrm{Re}(z) > 0$ by the Euler integral

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt.$$

Since with $x := \mathrm{Re}(z)$ one has $|t^{z-1}e^{-t}| = t^{x-1}e^{-t}$, the convergence of this integral follows from the corresponding fact in the real case (which we suppose known) and we have the estimate

$$|\Gamma(z)| \le \Gamma(\mathrm{Re}(z)) \quad \text{for } \mathrm{Re}(z) > 0.$$

Since the integrand depends holomorphically on z, it follows further that Γ is holomorphic in the halfplane $H(0) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. As in the real case one proves by partial integration the functional equation

$$z\Gamma(z) = \Gamma(z+1),$$

which together with $\Gamma(1) = 1$ shows that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}_0$. Applying the functional equation $n+1$ times yields

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdot\ldots\cdot(z+n)}.$$

The right hand side of this formula, which was derived for $\mathrm{Re}(z) > 0$, defines a meromorphic function in the halfplane $H(-n-1) = \{z \in \mathbb{C} : \mathrm{Re}(z) > -n-1\}$ having poles of first order at the points $z = -k$, $k = 0, 1, \ldots, n$. Therefore we can use this formula to continue the Gamma function analytically to a meromorphic function in the whole plane $\mathbb{C}$, with poles of first order at $z = -n$, $n \in \mathbb{N}_0$, and holomorphic elsewhere. From now on, by Gamma function we understand this meromorphic function in $\mathbb{C}$.

The Gamma function can be characterized axiomatically as follows:

9.2. Theorem. *Let F be a meromorphic function in $\mathbb{C}$ with the following properties:*

i) *F is holomorphic in the halfplane $H(0) = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.*

ii) *F satisfies the functional equation $zF(z) = F(z+1)$.*

iii) *F is bounded in the strip $\{z \in \mathbb{C} : 1 \le \mathrm{Re}(z) \le 2\}$.*

Then there exists a constant $c \in \mathbb{C}$ such that

$$F(z) = c\,\Gamma(z).$$

Proof. It is clear that Γ satisfies the properties i) to iii). We set $c := F(1)$ and

$$\Phi(z) := F(z) - c\,\Gamma(z).$$

Then Φ is also a function satisfying i) to iii) and $\Phi(1) = 0$. From the functional equation $\Phi(z) = \Phi(z+1)/z$ it follows that Φ is holomorphic at $z = 0$ and that Φ is bounded in the strip $\{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq 1\}$. Therefore the function

$$\varphi(z) := \Phi(z)\Phi(1-z)$$

is bounded in the same strip. We have

$$\varphi(z+1) = \Phi(z+1)\Phi(-z) = z\Phi(z)\Phi(-z) = -\Phi(z)\Phi(-z+1) = -\varphi(z).$$

From this it follows that φ is periodic with period 2 and bounded everywhere, hence holomorphic in $\mathbb{C}$. By the theorem of Liouville φ must be constant. Since $\varphi(1) = -\varphi(0)$, this constant is 0. The equation $0 = \Phi(z)\Phi(1-z)$ shows that also Φ is identically 0, but this means $F(z) = c\,\Gamma(z)$, q.e.d.

9.3. Theorem. a) *For every $z \in \mathbb{C} \setminus \{n \in \mathbb{Z} : n \leq 0\}$ we have*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdot\ldots\cdot(z+n)}$$

(Gauß representation of the Gamma function)

b) $1/\Gamma$ *is an entire holomorphic function with product representation*

$$\frac{1}{\Gamma(z)} = e^{Cz} z \prod_{n=1}^{\infty}\Big(1+\frac{z}{n}\Big)e^{-z/n}, \qquad (C = \text{Euler-Mascheroni constant}).$$

This product converges normally in $\mathbb{C}$.

Proof.

9.4. Lemma. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire holomorphic function and let $\rho, C, R_0 \in \mathbb{R}_+$ be non-negative constants such that*

$$\mathrm{Re}(f(z)) \leq C|z|^\rho \quad \text{for } |z| \geq R_0.$$

Then f is a polynomial of degree $\leq \rho$.

Note that no lower bound for $\mathrm{Re}(f(z))$ is required.

Proof. The Taylor series $f(z) = \sum_{n=0}^\infty a_n z^n$ converges for all $z \in \mathbb{C}$. Setting $z = Re^{it}$, we get Fourier series

$$f(Re^{it}) = \sum_{n=0}^\infty a_n R^n e^{int} \quad \text{and} \quad \overline{f(Re^{it})} = \sum_{n=0}^\infty \overline{a}_n R^n e^{-int},$$

hence

$$\mathrm{Re}(f(Re^{it})) = \mathrm{Re}(a_0) + \tfrac{1}{2}\sum_{n=1}^\infty R^n(a_n e^{int} + \overline{a}_n e^{-int}).$$

Multiplying this equation by e^{-ikt} and integrating from 0 to 2π yields

$$a_k = \frac{1}{\pi R^k} \int_0^{2\pi} \mathrm{Re}\left(f(Re^{it})\right) e^{-ikt} dt \quad \text{for } k > 0$$

and (for $k = 0$)

$$\mathrm{Re}(a_0) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(f(Re^{it})\right) dt.$$

The hypothesis on the growth of $\mathrm{Re}(f(z))$ implies

$$|\mathrm{Re}(f(z))| \le 2C|z|^\rho - \mathrm{Re}(f(z)) \quad \text{for } |z| \ge R_0,$$

(note this is true also if $\mathrm{Re}(f(z)) < 0$). Therefore we get the estimate

$$|a_k| \le \frac{1}{\pi R^k} \int_0^{2\pi} \left|\mathrm{Re}(f(re^{it}))\right| dt \le \frac{1}{R^k} \left(4CR^\rho - 2\mathrm{Re}(a_0)\right).$$

Letting $R \to \infty$, we see that $a_k = 0$ for $k > \rho$, q.e.d.

9.5. Theorem. *The Gamma function satisfies the following relations:*

a) $$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi},$$

b) $$\Gamma\Big(\frac{z}{2}\Big)\Gamma\Big(\frac{z+1}{2}\Big) = 2^{1-z}\sqrt{\pi}\,\Gamma(z).$$

Example. Setting $z = \frac{1}{2}$ in formula a) yields

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

The same result can also be obtained from formula b) for $z = 1$.

Proof. a) We first consider the meromorphic function

$$\Phi(z) := \Gamma(z)\Gamma(1-z).$$

It has poles of order 1 at the points $z = n$, $n \in \mathbb{Z}$, and is holomorphic elsewhere. It satisfies the relations

$$\Phi(z+1) = -\Phi(z) \quad \text{and} \quad \Phi(-z) = -\Phi(z).$$

Since $\Gamma(z)$ is bounded on $1 \le \mathrm{Re}(z) \le 2$ and

$$\Gamma(z) = \frac{\Gamma(1+z)}{z}, \quad \Gamma(1-z) = \frac{\Gamma(2-z)}{(1-z)},$$

it follows that Φ is bounded on the set

$$S_1 := \{z \in \mathbb{C} : 0 \le \mathrm{Re}(z) \le 1,\ |\mathrm{Im}(z)| \ge 1\}.$$

As $\sin(\pi z)$ has zeroes of order 1 at $z = n$, $n \in \mathbb{Z}$, the product

$$F(z) := \sin(\pi z)\Phi(z) = \sin(\pi z)\Gamma(z)\Gamma(1-z)$$

is holomorphic everywhere in $\mathbb{C}$ and without zeroes. We can write

$$F(z) = \frac{\sin(\pi z)}{z}\Gamma(1+z)\Gamma(1-z),$$

hence $F(0) = \pi$. Furthermore F is periodic with period 1 and an even function, i.e. $F(-z) = F(z)$. From the boundedness of Φ on S_1 we get an estimate

$$|F(z)| \le Ce^{\pi|z|} \quad \text{for } z \in S_1 \text{ and some constant } C > 0.$$

Since F is continuous and periodic, such an estimate holds in the whole plane $\mathbb{C}$. We can write F as $F(z) = e^{f(z)}$ with some holomorphic function $f : \mathbb{C} \to \mathbb{C}$. From $|F(z)| = e^{\mathrm{Re}(z)}$ we get an estimate

$$\mathrm{Re}(f) \le C'|z| \quad \text{for } |z| \ge R_0 \text{ and some constant } C' > 0.$$

By lemma 9.4, f must be a linear polynomial, hence

$$F(z) = e^{a+bz}, \quad (a, b \in \mathbb{C}).$$

Since F is an even function, we have $b = 0$, so the function F is a constant, which must be $F(0) = \pi$. This proves part a) of the theorem.

b) This is proved by applying theorem 9.2 to the function

$$F(z) := 2^z\,\Gamma\Big(\frac{z}{2}\Big)\Gamma\Big(\frac{z+1}{2}\Big).$$

...

9.6. Corollary (Sine product). *For all $z \in \mathbb{C}$ one has*

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\Big(1 - \frac{z^2}{n^2}\Big).$$

9.7. Corollary (Wallis product).

a) $$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{(2n)^2}{(2n-1)(2n+1)},$$

b) $$\sqrt{\pi} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{2^{2n}(n!)^2}{(2n)!}.$$

Proof. Formula a) follows directly from the sine product with $z = \frac{1}{2}$.
To prove formula b), we rewrite a) as

$$\begin{aligned}\pi &= 2 \lim_{n\to\infty} \prod_{k=1}^{n} \frac{(2k)^2}{(2k-1)(2k+1)} \\ &= \lim_{n\to\infty} \frac{2}{2n+1} \cdot \frac{2^{2n}(n!)^2}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot\ldots\cdot(2n-1)(2n-1)}\end{aligned}$$

hence

$$\begin{aligned}\sqrt{\pi} &= \lim_{n\to\infty} \sqrt{\frac{2}{2n+1}} \cdot \frac{2^n n!}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)} \\ &= \lim_{n\to\infty} \sqrt{\frac{2}{2n+1}} \cdot \frac{2^{2n}(n!)^2}{(2n)!}.\end{aligned}$$

Since $\lim\limits_{n\to\infty} \sqrt{2n}/\sqrt{2n+1} = 1$, the assertion follows.

9.8. Theorem (Stirling formula). *We have the following asymptotic relation*

$$n! \sim \sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n.$$

Proof. We apply the Euler-Maclaurin summation formula to

$$\log(n!) = \sum_{k=1}^{n} \log k$$

and obtain

$$\begin{aligned}\log(n!) &= \tfrac{1}{2}\log n + \int_1^n \log x\,dx + \int_1^n \frac{\mathrm{saw}(x)}{x}\,dx \\ &= \tfrac{1}{2}\log n + n(\log n - 1) + 1 + \int_1^n \frac{\mathrm{saw}(x)}{x}\,dx.\end{aligned}$$

Taking the exponential function of both sides we get

$$n! = \sqrt{n}\Big(\frac{n}{e}\Big)^n e^{\alpha_n},$$

where

$$\alpha_n = 1 + \int_1^n \frac{\mathrm{saw}(x)}{x}\,dx = 1 + \frac{B_2}{2}\cdot\frac{1}{x}\Big|_1^n + \int_1^n \frac{\widetilde{B}_2(x)}{2}\cdot\frac{1}{x^2}\,dx.$$

This last representation shows that

$$\alpha := \lim_{n\to\infty} \alpha_n = 1 + \int_1^\infty \frac{\operatorname{saw}(x)}{x}\,dx$$

exists and we have the asymptotic relation

$$n! \sim \sqrt{n}\Big(\frac{n}{e}\Big)^n e^\alpha.$$

It remains to prove that $e^\alpha = \sqrt{2\pi}$. This can be done as follows. Dividing the asymptotic relations

$$(n!)^2 \sim n\Big(\frac{n}{e}\Big)^{2n} e^{2\alpha} \qquad \text{and} \qquad (2n)! \sim \sqrt{2n}\Big(\frac{2n}{e}\Big)^{2n} e^\alpha$$

yields

$$e^\alpha = \lim_{n\to\infty} \frac{(n!)^2 e^{2n}}{n^{2n+1}} \cdot \frac{(2n)^{2n+1/2}}{(2n)!e^{2n}} = \lim_{n\to\infty} \frac{(n!)^2 \cdot 2^{2n+1/2}}{(2n)!\sqrt{n}} = \lim_{n\to\infty} \sqrt{\frac{2}{n}} \cdot \frac{2^{2n}(n!)^2}{(2n)!}.$$

Now corollary 9.7 shows $e^\alpha = \sqrt{2\pi}$, q.e.d.
For later use we note that we have hereby proved

$$1 + \int_1^\infty \frac{\operatorname{saw}(x)}{x}\,dx = \log\sqrt{2\pi}.$$

9.9. Theorem (Asymptotic expansion of the Gamma function). *For every integer $r \geq 1$ and every $z \in \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$ one has*

$$\begin{aligned}\log\Gamma(z) \;&=\; (z - \tfrac{1}{2})\log z - z + \log\sqrt{2\pi} + \sum_{k=1}^r \frac{B_{2k}}{(2k-1)2k} \cdot \frac{1}{z^{2k-1}} \\ &\quad + \int_0^\infty \frac{\widetilde{B}_{2r}(t)}{2r} \cdot \frac{1}{(z+t)^{2r}}\,dt.\end{aligned}$$

Here $\log\Gamma(z)$ and $\log z$ are those branches of the logarithm which take real values for positive real arguments.

Example. For $r = 5$, the value of the sum is

$$\sum_{k=1}^5 \frac{B_{2k}}{(2k-1)2k} \cdot \frac{1}{z^{2k-1}} = \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \frac{1}{1680z^7} + \frac{1}{1188z^9}$$

Proof. We use the Gauß representation of the Gamma function (theorem 9.3.a) and get

$$\log\Gamma(z) = \lim_{n\to\infty}\Big(z\log n + \sum_{k=1}^n \log k - \sum_{k=0}^n \log(z+k)\Big).$$

By Euler-Maclaurin (theorem 5.2)

$$\begin{aligned}\sum_{k=1}^{n} \log k &= \tfrac{1}{2}\log n + \int_1^n \log t\,dt + \int_1^n \frac{\operatorname{saw}(t)}{t}dt \\ &= \tfrac{1}{2}\log n + n\log n - n + 1 + \int_1^n \frac{\operatorname{saw}(t)}{t}dt\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} \log(z+k) &= \tfrac{1}{2}(\log z + \log(z+n)) + \int_0^n \log(z+t)dt + \int_0^n \frac{\operatorname{saw}(t)}{z+t}dt \\ &= \tfrac{1}{2}(\log z + \log(z+n)) + (z+n)\log(z+n) - z\log z - n \\ &\qquad + \int_0^n \frac{\operatorname{saw}(t)}{z+t}dt.\end{aligned}$$

Therefore

$$\begin{aligned}&z\log n + \sum_{k=1}^{n}\log k - \sum_{k=0}^{n}\log(z+k) \\ &\quad = (z-\tfrac{1}{2})\log z - (z+n+\tfrac{1}{2})\log\left(1+\frac{z}{n}\right) + 1 \\ &\qquad + \int_1^n \frac{\operatorname{saw}(t)}{t}dt - \int_0^n \frac{\operatorname{saw}(t)}{z+t}dt.\end{aligned}$$

Since

$$\lim_{n\to\infty}(z+n+\tfrac{1}{2})\log\left(1+\frac{z}{n}\right) = z$$

and

$$\lim_{n\to\infty}\left(1+\int_1^n \frac{\operatorname{saw}(t)}{t}dt\right) = \log\sqrt{2\pi} \quad \text{(see above)},$$

we get

$$\log\Gamma(z) = (z-\tfrac{1}{2})\log z - z + \log\sqrt{2\pi} - \int_0^\infty \frac{\operatorname{saw}(t)}{z+t}dt.$$

The rest is proved as in theorem 5.11.