8. Primes in Arithmetic Progressions

8.1. Definition (Dirichlet density). For any subset $A \subset \mathbb{P}$ of the set $\mathbb{P}$ of all primes, we define the function

$$P_A(s) := \sum_{p \in A} \frac{1}{p^s}.$$

The sum converges at least for $\mathrm{Re}(s) > 1$ and defines a holomorphic function in the halfplane $H(1) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$. For $A = \mathbb{P}$ we get the prime zeta function $P(s)$ already discussed in (4.7). If the limit

$$\delta_{\mathrm{Dir}}(A) := \lim_{\sigma \searrow 1} \frac{P_A(\sigma)}{P(\sigma)}$$

exists, it is called the *Dirichlet density* or *analytic density* of the set A. It is clear that, if the Dirichlet density of A exists, one has

$$0 \le \delta_{\mathrm{Dir}}(A) \le 1.$$

The Dirichlet density of the set of all primes is 1, and any finite set of primes has density 0. Hence $\delta_{\mathrm{Dir}}(A) > 0$ implies that A is infinite.
An equivalent definition of the Dirichlet density is

$$\delta_{\mathrm{Dir}}(A) = \lim_{\sigma \searrow 1} P_A(\sigma) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big).$$

This comes from the fact that

$$\lim_{\sigma \searrow 1} P(\sigma)/\log\zeta(\sigma) = 1$$

by theorem 4.7, and

$$\lim_{\sigma \searrow 1} \log\zeta(\sigma) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = 1,$$

since $\zeta(s) = 1/(s-1) + $ (holomorphic function).

8.2. Arithmetic progressions. Let m, a be integers, $m \ge 2$. The set of all $n \in \mathbb{N}_1$ with

$$n \equiv a \bmod m$$

is called an arithmetic progression. We want to study the distribution of primes in arithmetic progressions. Clearly if $\gcd(a, m) > 1$, there exist only finitely many primes in the arithmetic progression of numbers congruent $a \bmod m$. So suppose $\gcd(a, m) = 1$. Dirichlet has proved that there exist infinitely many primes $p \equiv a \bmod m$, more

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precisely: The set of all such primes has Dirichlet density $1/\varphi(m)$, which means that the Dirichlet density of primes in all arithmetic progressions $a \bmod m$, $\gcd(a,m)=1$, is the same. To prove this, we have, according to definition 8.1, to study the functions

$$P_{a,m}(s) := \sum_{p \equiv a \bmod m} \frac{1}{p^s},$$

where the sum is extended over all primes $\equiv a \bmod m$. It was Dirichlet's idea to use instead the functions

$$P(s,\chi) := \sum_{p\in\mathbb{P}} \frac{\chi(p)}{p^s},$$

where $\chi : \mathbb{N}_1 \to \mathbb{C}$ is a Dirichlet character modulo m. These functions were already introduced in theorem 7.7. The relation between the functions $P_{a,m}(s)$ and $P(s,\chi)$ is given by the following lemma.

8.3. Lemma. *Let m be an integer ≥ 2 and a an integer coprime to m. Then we have for all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 1$

$$P_{a,m}(s) = \frac{1}{\varphi(m)} \sum_{\chi} \overline{\chi}(a) P(s,\chi).$$

Here the sum is extended over all Dirichlet charcters χ modulo m and $\overline{\chi}(a)$ denotes the complex conjugate of $\chi(a)$.

Proof. We have

$$\sum_{\chi} \overline{\chi}(a) P(s,\chi) = \sum_{p\in\mathbb{P}} \Big(\sum_{\chi} \overline{\chi}(a)\chi(p) \Big) \cdot \frac{1}{p^s} = \sum_{p\in\mathbb{P}} \frac{\alpha_p}{p^s},$$

where

$$\alpha_p := \sum_{\chi} \overline{\chi}(a)\chi(p).$$

Since a is coprime to m, there exists an integer b with $ab \equiv 1 \bmod m$, hence $\chi(a)\chi(b) = 1$. On the other hand $|\chi(a)| = 1$, which implies $\chi(b) = \overline{\chi}(a)$. Therefore by theorem 7.3.b)

$$\alpha_p = \sum_{\chi} \chi(b)\chi(p) = \sum_{\chi} \chi(bp) = \begin{cases} \varphi(m) & \text{if } bp \equiv 1 \bmod m, \\ 0 & \text{otherwise.} \end{cases}$$

But $bp \equiv 1 \bmod m$ is equivalent to $p \equiv a \bmod m$, hence

$$\sum_{p\in\mathbb{P}} \frac{\alpha_p}{p^s} = \varphi(m) \sum_{p \equiv a \bmod m} \frac{1}{p^s},$$

which proves the lemma.

In the proof of the Dirichlet theorem on primes in arithmetic progressions, the following theorem plays an essential role.

8.4. Theorem. *Let m be an integer ≥ 2 and χ a non-principal Dirichlet character modulo m. Then*

$$L(1, \chi) \neq 0.$$

Recall that for a non-principal character χ the function $L(s, \chi)$ is holomorphic for $\mathrm{Re}(s) > 0$ (theorem 7.6.c).

Example. For the non-principal character χ_1 modulo 4 one has (cf. 7.4)

$$L(1, \chi_1) = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} \pm \ldots = \frac{\pi}{4}.$$

Before we prove this theorem, we show how Dirichlet's theorem can be derived from it.

8.5. Theorem (Dirichlet). *Let a, m be coprime integers, $m \geq 2$. Then the set of all primes $p \equiv a \bmod m$ has Dirichlet density $1/\varphi(m)$.*

Proof. For the principal Dirichlet character χ_{0m} it follows from theorem 7.6.b) that

$$\lim_{\sigma \searrow 1} \log L(\sigma, \chi_{0m}) / \log \zeta(\sigma) = \lim_{\sigma \searrow 1} \log L(\sigma, \chi_{0m}) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = 1.$$

On the other hand, if χ is a non-principal character, then we have by theorem 8.4

$$\lim_{\sigma \searrow 1} \log L(\sigma, \chi) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = 0.$$

By theorem 7.7 this implies

$$\lim_{\sigma \searrow 1} P(\sigma, \chi_{0m}) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = 1$$

and

$$\lim_{\sigma \searrow 1} P(\sigma, \chi) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = 0$$

for all non-principal characters χ. Therefore

$$\lim_{\sigma \searrow 1} \Big(\sum_{\chi} \overline{\chi}(a) P(\sigma, \chi)\Big) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = \overline{\chi}_{0m}(a) = 1.$$

Now using lemma 8.3 we get

$$\lim_{\sigma \searrow 1} P_{a,m}(\sigma) \Big/ \log\Big(\frac{1}{\sigma - 1}\Big) = \frac{1}{\varphi(m)},$$

which proves our theorem.

8.6. *Proof of theorem 8.4.* We have to show that

$$L(1,\chi) \neq 0$$

for every non-principal Dirichlet character χ modulo m.
Assume to the contrary that there exists at least one non-principal character χ with $L(1,\chi) = 0$. We define the function

$$\zeta_m(s) := \prod_{\chi} L(s,\chi),$$

where the product is extended over *all* Dirichlet characters modulo m. For the principal character the function $L(s,\chi_{0m})$ has a pole of order 1 at $s = 1$. This pole is canceled by the assumed zero of one of the functions $L(s,\chi)$, $\chi \neq \chi_{0m}$. Therefore, under the assumption, ζ_m would be holomorphic everywhere in the halfplane $H(0) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$. We will show that this leads to a contradiction.
Using the Euler product for the L-functions (theorem 7.6), we get

$$\zeta_m(s) = \prod_{\chi}\prod_{p\in\mathbb{P}} \frac{1}{1-\chi(p)p^{-s}} = \prod_{p\in\mathbb{P}} \frac{1}{\prod_\chi(1-\chi(p)p^{-s})}.$$

By lemma 8.7 below, for every $p \nmid m$ there exist integers $f(p), g(p) \geq 1$ with $f(p)g(p) = \varphi(m)$ such that

$$\prod_{\chi}(1-\chi(p)p^{-s}) = (1-p^{-f(p)s})^{g(p)}.$$

Therefore

$$\frac{1}{\prod_\chi(1-\chi(p)p^{-s})} = \Big(\sum_{k=0}^{\infty} \frac{1}{p^{f(p)ks}}\Big)^{g(p)}$$

is a Dirichlet series with non-negative coefficients and we have

$$\Big(\sum_{k=0}^{\infty} \frac{1}{p^{f(p)ks}}\Big)^{g(p)} \succ \sum_{k=0}^{\infty} \frac{1}{p^{\varphi(m)ks}},$$

where the relation $\sum_n a_n/n^s \succ \sum_n b_n/n^s$ between two Dirichlet series is defined as $a_n \geq b_n$ for all n. It follows that $\zeta_m(s)$ is a Dirichlet series with non-negative coefficients and

$$\zeta_m(s) \succ \prod_{p\nmid m}\Big(\sum_{k=0}^{\infty} \frac{1}{p^{\varphi(m)ks}}\Big) = \sum_{\gcd(n,m)=1} \frac{1}{n^{\varphi(m)s}}.$$

The last Dirichlet series has abscissa of absolute convergence $= 1/\varphi(m)$. Therefore $\sigma_a(\zeta_m) \geq 1/\varphi(m)$. But by the theorem of Landau (6.8) this contradicts the assumption that ζ_m is holomorphic in the halfplane $H(0)$. Therefore the assumption is false, which proves $L(1,\chi) \neq 0$ for all non-principal characters χ.

8.7. Lemma. *Let G be a finite abelian group of order r and let $g \in G$ be an element of order $k \mid r$. Then we have the following identity in the polynomial ring $\mathbb{C}[T]$*

$$\prod_{\chi \in \widehat{G}} (1 - \chi(g)T) = \big(1 - T^k\big)^{r/k}.$$

Proof. Let $H \subset G$ be the subgroup generated by the element g. H is a cyclic group of order k. For every character $\chi \in \widehat{G}$, the restriction $\chi \mid H$ is a character of H. Two characters $\chi_1, \chi_2 \in \widehat{G}$ have the same restriction to H iff the character $\chi := \chi_1\chi_2^{-1}$ is identically 1 on H, which implies that χ induces a character on the quotient group G/H. Since G/H has r/k elements, there can be at most r/k characters of G which restrict to the unit character on H. This means that the restriction of the r characters of G yield at least k different characters of H. But we know that there are exactly k characters of H. Hence every character ψ of H is the restriction of a character of G and there are exactly r/k characters of G which restrict to ψ. Now

$$\prod_{\psi \in \widehat{H}} (1 - \psi(g)T) = \prod_{\nu=0}^{k-1} (1 - e^{2\pi i\nu/k}T) = 1 - T^k$$

and

$$\prod_{\chi \in \widehat{G}} (1 - \chi(g)T) = \Big(\prod_{\psi \in \widehat{H}} (1 - \psi(g)T)\Big)^{r/k} = \big(1 - T^k\big)^{r/k}, \quad \text{q.e.d.}$$