6. Dirichlet Series

6.1. Definition. A *Dirichlet series* is a series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad (s \in \mathbb{C}),$$

where $(a_n)_{n\geqslant 1}$ is an arbitrary sequence of complex numbers.

The *abscissa of absolute convergence* of this series is defined as

$$\sigma_a := \sigma_a(f) := \inf\{\sigma \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty\} \in \mathbb{R} \cup \{\pm\infty\}.$$

If $\sum_{n=1}^{\infty}(|a_n|/n^\sigma)$ does not converge for any $\sigma \in \mathbb{R}$, then $\sigma_a = +\infty$, if it converges for all $\sigma \in \mathbb{R}$, then $\sigma_a = -\infty$.

An analogous argument as in the case of the zeta function shows that a Dirichlet series with abscissa of absolute convergence σ_a converges absolutely and uniformly in every halfplane $\overline{H(\sigma)}$, $\sigma > \sigma_a$.

Example. The Dirichlet series

$$g(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

has $\sigma_a(g) = 1$. We will see however that the series converges for every $s \in H(0)$. Of course the convergence is only conditional and not absolute if $0 < \mathrm{Re}(s) \leq 1$.

We need some preparations.

6.2. Lemma (Abel summation). *Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 1}$ be two sequences of complex numbers and set*

$$A_n := \sum_{k=1}^{n} a_k, \qquad A_0 = 0 \text{ (empty sum)}.$$

Then we have for all $n \geq m \geq 1$

$$\sum_{k=m}^{n} a_k b_k = A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k).$$

Remark. This can be viewed as an analogon of the formula for partial integration

$$\int_a^b F'(x) g(x) dx = F(b)g(b) - F(a)g(a) - \int_a^b F(x) g'(x) dx.$$

Proof.

$$\begin{aligned}\sum_{k=m}^{n} a_k b_k &= \sum_{k=m}^{n}(A_k - A_{k-1})b_k = \sum_{k=m}^{n} A_k b_k - \sum_{k=m-1}^{n-1} A_k b_{k+1}\\ &= A_n b_n + \sum_{k=m}^{n-1} A_k b_k - \sum_{k=m}^{n-1} A_k b_{k+1} - A_{m-1} b_m\\ &= A_n b_n - A_{m-1} b_m - \sum_{k=m}^{n-1} A_k (b_{k+1} - b_k), \quad \text{q.e.d.}\end{aligned}$$

6.3. Lemma. *Let $s \in \mathbb{C}$ with $\sigma := \mathrm{Re}(s) > 0$. Then we have for all $m, n \geq 1$*

$$\left|\frac{1}{n^s} - \frac{1}{m^s}\right| \;\leq\; \frac{|s|}{\sigma} \cdot \left|\frac{1}{n^\sigma} - \frac{1}{m^\sigma}\right|.$$

Proof. We may assume $n \geq m$. Since $\dfrac{d}{dx}\Big(\dfrac{1}{x^s}\Big) = -s \cdot \dfrac{1}{x^{s+1}}$,

$$-s \int_m^n \frac{dx}{x^{s+1}} = \frac{1}{n^s} - \frac{1}{m^s}\,.$$

Taking the absolute values, we get the estimate

$$\left|\frac{1}{n^s} - \frac{1}{m^s}\right| \leq |s| \int_m^n \frac{dx}{x^{\sigma+1}} = \frac{|s|}{\sigma} \cdot \left|\frac{1}{n^\sigma} - \frac{1}{m^\sigma}\right|, \quad \text{q.e.d.}$$

Remark. For $s_0 \in \mathbb{C}$ and an angle α with $0 < \alpha < \pi/2$, we define the *angular region*

$$\mathrm{Ang}(s_0, \alpha) := \{s_0 + re^{i\phi} : r \geq 0 \text{ and } |\phi| \leq \alpha\}.$$

For any $s \in \mathrm{Ang}(s_0, \alpha) \smallsetminus \{s_0\}$ we have

$$\frac{|s - s_0|}{\mathrm{Re}(s - s_0)} = \frac{1}{\cos\phi} \leq \frac{1}{\cos\alpha}\,,$$

hence the estimate in lemma 6.3 can be rewritten as

$$\left|\frac{1}{n^s} - \frac{1}{m^s}\right| \;\leq\; \frac{1}{\cos\alpha} \cdot \left|\frac{1}{n^\sigma} - \frac{1}{m^\sigma}\right| \quad \text{for all } s \in \mathrm{Ang}(0, \alpha).$$

6.4. Theorem. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series such that for some $s_0 \in \mathbb{C}$ *the partial sums* $\sum\limits_{n=1}^{N} \dfrac{a_n}{n^{s_0}}$ *are bounded for* $N \to \infty$*. Then the Dirichlet series converges for every* $s \in \mathbb{C}$ *with*

$$\mathrm{Re}(s) > \sigma_0 := \mathrm{Re}(s_0).$$

The convergence is uniform on every compact subset

$$K \subset H(\sigma_0) = \{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_0\}.$$

Hence f *is a holomorphic function in* $H(\sigma_0)$*.*

Proof. Since

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s_0}} \cdot \frac{a_n}{n^{s-s_0}} = \sum_{n=1}^{\infty} \frac{\tilde{a}_n}{n^{s-s_0}} \quad \text{where } \tilde{a}_n := \frac{a_n}{n^{s_0}},$$

we may suppose without loss of generality that $s_0 = 0$. By hypothesis there exists a constant $C_1 > 0$ such that

$$\Big|\sum_{n=1}^{N} a_n\Big| \le C_1 \quad \text{for all } N \in \mathbb{N}.$$

The compact set K is contained in some angular region $\mathrm{Ang}(0, \alpha)$ with $0 < \alpha < \pi/2$. We define

$$C_\alpha := \frac{1}{\cos\alpha} \quad \text{and} \quad \sigma_* := \inf\{\mathrm{Re}(s) : s \in K\} > 0.$$

Now we apply the Abel summation lemma 6.2 to the sum $\sum a_n \cdot (1/n^s)$, $s \in K$. Setting $A_N := \sum_{n=1}^{N} a_n$, we get for $N \ge M \ge 1$

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = A_N \frac{1}{N^s} - A_{M-1} \frac{1}{M^s} + \sum_{n=M}^{N-1} A_n \Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big).$$

This leads to the estimate (with $\sigma = \mathrm{Re}(s)$)

$$\begin{aligned}
\Big|\sum_{n=M}^{N} \frac{a_n}{n^s}\Big| &\le 2C_1 \Big|\frac{1}{M^s}\Big| + C_1 \sum_{n=M}^{N-1} \Big|\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big| \\
&\le 2C_1 \frac{1}{M^\sigma} + C_1 C_\alpha \sum_{n=M}^{N-1} \Big(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\Big) \\
&= 2C_1 \frac{1}{M^\sigma} + C_1 C_\alpha \Big(\frac{1}{M^\sigma} - \frac{1}{N^\sigma}\Big) \\
&\le \frac{C_1}{M^\sigma}\Big(2 + C_\alpha\Big) \le \frac{C_1(2 + C_\alpha)}{M^{\sigma_*}}.
\end{aligned}$$

This becomes arbitrarily small if M is sufficently large. This implies the asserted uniform convergence on K of the Dirichlet series.

6.5. Theorem. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series which converges for some $s_0 \in \mathbb{C}$. *Then the series converges uniformly in every angular region* $\mathrm{Ang}(s_0, \alpha)$, $0 < \alpha < \pi/2$. *In particular*

$$\lim_{s \to s_0} f(s) = f(s_0),$$

when s *approaches* s_0 *within an angular region* $\mathrm{Ang}(s_0, \alpha)$.

Proof. As in the proof of theorem 6.4 we may suppose $s_0 = 0$. Set $C_\alpha := 1/\cos\alpha$. Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} a_n$ converges, there exists an $n_0 \in \mathbb{N}$, such that

$$\Big|\sum_{n=M}^{N} a_n\Big| < \varepsilon_1 := \frac{\varepsilon}{1 + C_\alpha} \quad \text{for all } N \geq M \geq n_0.$$

With $A_{Mn} := \sum_{k=M}^{n} a_k$, $A_{M,M-1} = 0$, we have by the Abel summation formula

$$\sum_{n=M}^{N} \frac{a_n}{n^s} = A_{MN} \frac{1}{N^s} + \sum_{n=M}^{N-1} A_{Mn} \Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big).$$

From this, we get for all $s \in \mathrm{Ang}(0, \alpha)$, $\sigma := \mathrm{Re}(s)$, and $N \geq M \geq n_0$ the estimate

$$\begin{aligned}\Big|\sum_{n=M}^{N} \frac{a_n}{n^s}\Big| &\leq \varepsilon_1 \frac{1}{|N^s|} + \varepsilon_1 \sum_{n=M}^{N-1} \Big|\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big| \\ &\leq \varepsilon_1 + \varepsilon_1 C_\alpha \sum_{n=M}^{N-1} \Big(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\Big) \\ &= \varepsilon_1 + \varepsilon_1 C_\alpha \Big(\frac{1}{M^\sigma} - \frac{1}{N^\sigma}\Big) \leq \varepsilon_1 + \varepsilon_1 C_\alpha = \varepsilon.\end{aligned}$$

This shows the uniform convergence of the Dirichlet series in $\mathrm{Ang}(0, \alpha)$. Therefore f is continuous in $\mathrm{Ang}(0, \alpha)$, which implies the last assertion of the theorem.

6.6. Definition. Let $f(s) = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series. The *abscissa of convergence* of f is defined by

$$\sigma_c := \sigma_c(f) := \inf \Big\{\mathrm{Re}(s) : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges } \Big\}.$$

By theorem 6.4 this is the same as

$$\sigma_c = \inf\{\mathrm{Re}(s) : \sum_{n=1}^{N} \frac{a_n}{n^s} \text{ is bounded for } N \to \infty\}$$

and it follows that the series converges to a holomorphic function in the halfplane $H(\sigma_c)$.

Examples. Consider the three Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad g(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \qquad \frac{1}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We have $\sigma_a(\zeta) = \sigma_a(g) = \sigma_a(1/\zeta) = 1$. Clearly $\sigma_c(\zeta) = 1$ and $\sigma_c(g) = 0$, since the partial sums $\sum_{n=1}^{N}(-1)^{n-1}$ are bounded. The abscissa of convergence $\sigma_c(1/\zeta)$ is not known; of course $\sigma_c(1/\zeta) \le 1$. One conjectures that $\sigma_c(1/\zeta) = \frac{1}{2}$, which is equivalent to the *Riemann Hypothesis*, which we will discuss in a later chapter.

Remark. Multiplying the zeta series by 2^{-s} yields $2^{-s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$. Hence

$$g(s) = (1 - 2^{1-s})\zeta(s).$$

6.7. Theorem. *If the Dirichlet series* $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *has a finite abscissa of convergence* σ_c*, then for the abscissa of absolute convergence* σ_a *the following estimate holds:*

$$\sigma_c \le \sigma_a \le \sigma_c + 1.$$

Proof. Without loss of generality we may suppose $\sigma_c = 0$. Then $\sum_{n=1}^{\infty} \frac{a_n}{n^\varepsilon}$ converges for every $\varepsilon > 0$. We have to show that

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_*}} < \infty \quad \text{for all } \sigma_* > 1.$$

To see this, write $\sigma_* = 1 + 2\varepsilon$, $\varepsilon > 0$. Then

$$\frac{|a_n|}{n^{\sigma_*}} = \frac{|a_n|}{n^\varepsilon} \cdot \frac{1}{n^{1+\varepsilon}}$$

Since $|a_n|/n^\varepsilon$ is bounded for $n \to \infty$ and $\sum_{n=1}^{\infty} 1/n^{1+\varepsilon} < \infty$, the assertion follows.

Remarks. a) It can be easily seen that $\sigma_c = -\infty$ implies $\sigma_a = -\infty$.

b) The above examples show that the cases $\sigma_a = \sigma_c$ and $\sigma_a = \sigma_c + 1$ do actually occur.

c) That σ_a and σ_c may be different is quite surprising if one looks at the situation for power series: If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$, it converges absolutely for every z with $|z| < |z_0|$.

6.8. Theorem (Landau). *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with non-negative coefficients $a_n \geq 0$ and finite abscissa of absolute convergence $\sigma_a \in \mathbb{R}$. Then the function f, which is holomorphic in the halfplane $H(\sigma_a)$, cannot be continued analytically as a holomorphic function to any neighborhood of σ_a.

Proof. Assume to the contrary that there exists a small open disk D around σ_a such that f can be analytically continued to a holomorphic function in $H(\sigma_a) \cup D$, which we denote again by f. Then the Taylor series of f at the point $\sigma_1 := \sigma_a + 1$ has radius of convergence > 1. Since

$$f^{(k)}(\sigma_1) = \sum_{n=1}^{\infty} \frac{(-\log n)^k a_n}{n^{\sigma_1}},$$

the Taylor series has the form

$$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\sigma_1)}{k!}(s - \sigma_1)^k = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-\log n)^k a_n}{k!\, n^{\sigma_1}}(s - \sigma_1)^k.$$

By hypothesis there exists a real $\sigma < \sigma_a$ such that the Taylor series converges for $s = \sigma$. We have

$$f(\sigma) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(\log n)^k a_n (\sigma_1 - \sigma)^k}{k!\, n^{\sigma_1}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(\log n)^k (\sigma_1 - \sigma)^k}{k!} \cdot \frac{a_n}{n^{\sigma_1}},$$

where the reordering is allowed since all terms are non-negative. Now

$$\sum_{k=0}^{\infty} \frac{(\log n)^k (\sigma_1 - \sigma)^k}{k!} = e^{(\log n)(\sigma_1 - \sigma)} = \frac{1}{n^{\sigma - \sigma_1}},$$

hence we have a convergent series

$$f(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma - \sigma_1}} \cdot \frac{a_n}{n^{\sigma_1}} = \sum_{n=1}^{\infty} \frac{a_n}{n^{\sigma}}.$$

Thus the abscissa of absolute convergence is $\leq \sigma < \sigma_a$, a contradiction. Hence the assumption is false, which proves the theorem.

6.9. Theorem (Identity theorem for Dirichlet series). *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{and} \quad g(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

be two Dirichlet series that converge in a common halfplane $H(\sigma_0)$. If there exists a sequence $s_\nu \in H(\sigma_0)$, $\nu \in \mathbb{N}_1$, with $\lim_{\nu\to\infty} \mathrm{Re}(s_\nu) = \infty$ and

$$f(s_\nu) = g(s_\nu) \quad \text{for all } \nu \geq 1,$$

then $a_n = b_n$ for all $n \geq 1$.

Proof. Passing to the difference $f - g$ shows that it suffices to prove the theorem for the case where g is identically zero. So we suppose that

$$f(s_\nu) = 0 \quad \text{for all } \nu \geq 1.$$

If not all $a_n = 0$, then there exists a minimal k such that $a_k \neq 0$. We have

$$f(s) = \frac{1}{k^s}\Big(a_k + \sum_{n>k} \frac{a_n}{(n/k)^s}\Big).$$

It suffices to show that there exists a $\sigma_* \in \mathbb{R}$ such that

$$\Big|\sum_{n>k} \frac{a_n}{(n/k)^s}\Big| \leq \frac{|a_k|}{2} \quad \text{for all } s \text{ with } \mathrm{Re}(s) \geq \sigma_*,$$

for this would imply $f(s) \neq 0$ for $\mathrm{Re}(s) \geq \sigma_*$, contradicting $f(s_\nu) = 0$ for all ν. The sum $\sum_{n>k} \frac{a_n}{(n/k)^{\sigma'}}$ converges absolutely for some $\sigma' \in \mathbb{R}$. Therefore we can find an $M \geq k$ such that

$$\sum_{n>M} \frac{|a_n|}{(n/k)^{\sigma'}} \leq \frac{|a_k|}{4}.$$

Further there exists a $\sigma'' \in \mathbb{R}$ such that

$$\sum_{k<n\leqslant M} \frac{|a_n|}{(n/k)^{\sigma''}} \leq \frac{|a_k|}{4}.$$

Combining the last two estimates shows

$$\Big|\sum_{n>k} \frac{a_n}{(n/k)^s}\Big| \leq \frac{|a_k|}{2} \quad \text{for all } s \text{ with } \mathrm{Re}(s) \geq \max(\sigma', \sigma''), \quad \text{q.e.d.}$$

Remark. A similar theorem is not true for arbitrary holomorphic functions in halfplanes. For example, the sine function satisfies

$$\sin(\pi n) = 0 \quad \text{for all integers } n,$$

without being identically zero. This shows also that not every function holomorphic in a halfplane $H(\sigma)$ can be expanded in a Dirichlet series.

6.10. Theorem. *Let $a, b : \mathbb{N}_1 \to \mathbb{C}$ be two arithmetical functions such that the Dirichlet series*

$$f(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \quad \textit{and} \quad g(s) := \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$$

converge absolutely in a common halfplane $H(\sigma_0)$. Then we have for the product

$$f(s)g(s) = \sum_{n=1}^{\infty} \frac{(a*b)(n)}{n^s}.$$

This Dirichlet series converges absolutely in $H(\sigma_0)$.

Proof. Since the series for $f(s)$ and $g(s)$ converge absolutely for $s \in H(\sigma_0)$, they can be multiplied term by term

$$\begin{aligned} f(s)g(s) &= \sum_{k=1}^{\infty} \frac{a(k)}{k^s} \sum_{\ell=1}^{\infty} \frac{b(\ell)}{\ell^s} = \sum_{k,\ell \geqslant 1} a(k)b(\ell) \frac{1}{k^s \ell^s} \\ &= \sum_{n=1}^{\infty} \sum_{k\ell=n} a(k)b(\ell) \frac{1}{(k\ell)^s} = \sum_{n=1}^{\infty} \frac{(a*b)(n)}{n^s}, \end{aligned}$$

and the product series converges absolutely, q.e.d.

Examples. i) The zeta function $\zeta(s) = \sum\limits_{n=1}^{\infty} \frac{1}{n^s}$ is the Dirichlet series associated to the constant arithmetical function $u(n) = 1$. Since $u * \mu = \delta_1$, it follows

$$\Big(\sum_{n=1}^{\infty} \frac{1}{n^s}\Big)\Big(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}\Big) = \sum_{n=1}^{\infty} \frac{\delta_1(n)}{n^s} = 1,$$

which gives a new proof of

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad \text{(cf. theorem 4.5).}$$

ii) The Dirichlet series associated to the identity map $\iota : \mathbb{N}_1 \to \mathbb{N}_1$ is

$$\sum_{n=1}^{\infty} \frac{n}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} = \zeta(s-1),$$

which converges absolutely for $\mathrm{Re}(s) > 2$. For the divisor sum function σ we have $u * \iota = \sigma$, cf. (3.15.iii), which implies

$$\zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 2.$$

iii) In a similar way, the formula $\varphi = \mu * \iota$ for the Euler phi function, cf. (3.15.i), yields

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \quad \text{for } \mathrm{Re}(s) > 2.$$

6.11. Theorem (Euler product for Dirichlet series). *Let* $a : \mathbb{N}_1 \to \mathbb{C}$ *be a* multiplicative *arithmetical function such that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

has abscissa of absolute convergence $\sigma_a < \infty$.
a) *Then we have in* $H(\sigma_a)$ *the product representation*

$$f(s) = \prod_{p\in\mathbb{P}} \Big(\sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}} \Big) = \prod_{p\in\mathbb{P}} \Big(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \cdots \Big),$$

where the product is extended over the set $\mathbb{P}$ *of all primes.*
b) *If the arithmetical function* a *is* completely *multiplicative, this can be simplified to*

$$f(s) = \prod_{p\in\mathbb{P}} \Big(1 - \frac{a(p)}{p^s} \Big)^{-1}.$$

Proof. Let $\mathcal{P} \subset \mathbb{P}$ be a finite set of primes and $\mathbb{N}(\mathcal{P})$ the set of all positive integers whose prime decomposition contains only primes from the set $\mathcal{P}$. Since a is multiplicative, we have for an integer n with prime decomposition $n = p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_r^{k_r}$

$$a(n) = a(p_1^{k_1}) a(p_2^{k_2}) \cdot \ldots \cdot a(p_r^{k_r}).$$

It follows by multiplying the infinite series term by term that

$$\prod_{p\in\mathcal{P}} \Big(1 + \frac{a(p)}{p^s} + \frac{a(p^2)}{p^{2s}} + \frac{a(p^3)}{p^{3s}} + \cdots \Big) = \sum_{n\in\mathbb{N}(\mathcal{P})} \frac{a(n)}{n^s}.$$

Letting $\mathcal{P} = \mathcal{P}_m$ be set of all primes $\le m$ and passing to the limit $m \to \infty$, we obtain part a) the theorem.

If a is completely multiplicative, then $a(p^k) = a(p)^k$, hence

$$\sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}} = \sum_{k=0}^{\infty} \Big(\frac{a(p)}{p^s}\Big)^k = \Big(1 - \frac{a(p)}{p^s}\Big)^{-1},$$

proving part b).

Examples. i) The Euler product for the zeta function

$$\zeta(s) = \prod_{p\in\mathbb{P}} \Big(1 - \frac{1}{p^s}\Big)^{-1}$$

is a special case of this theorem.

ii) Since $\mu(p) = -1$ and $\mu(p^k) = 0$ for $k \geq 2$, the formula for the inverse of the zeta function

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p\in\mathbb{P}} \Big(1 - \frac{1}{p^s}\Big) = \frac{1}{\zeta(s)}$$

also follows from this theorem.