5. The Euler-Maclaurin Summation Formula

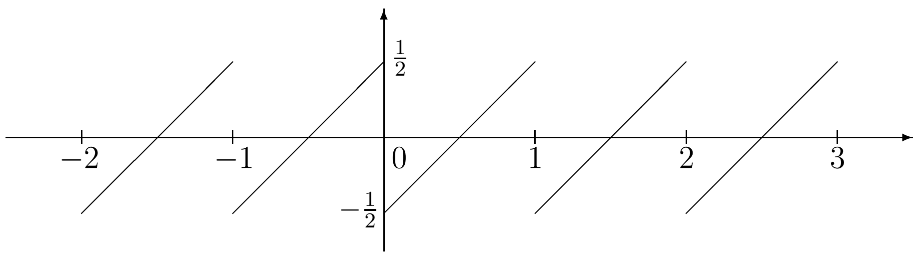
5.1. We define a periodic function

$$\mathrm{saw} : \mathbb{R} \longrightarrow \mathbb{R}$$

with period 1 by

$$\mathrm{saw}(x) := x - \lfloor x \rfloor - \tfrac{1}{2}$$

This is a kind of sawtooth function, see figure.



With this function, we can state a first form of the Euler-Maclaurin summation formula. This formula shows how a sum can be approximated by an integral and gives an exact error term.

5.2. Theorem (Euler-Maclaurin I). *Let x_0 be a real number and $f : [x_0, \infty[\to \mathbb{C}$ a continuously differentiable function. Then we have for all integers $n \geq m \geq x_0$*

$$\sum_{k=m}^{n} f(k) = \tfrac{1}{2}(f(m) + f(n)) + \int_m^n f(x)dx + \int_m^n \mathrm{saw}(x) f'(x) dx.$$

Proof. We have

$$\sum_{k=m}^{n} f(k) - \tfrac{1}{2}(f(m) + f(n)) = \sum_{k=m}^{n-1} \tfrac{1}{2}(f(k) + f(k+1)).$$

On the other hand we get by partial integration

$$\begin{aligned}
\int_k^{k+1} \mathrm{saw}(x) f'(x) dx &= \int_k^{k+1} (x - k - \tfrac{1}{2}) f'(x) dx \\
&= (x - k - \tfrac{1}{2}) f(x) \Big|_k^{k+1} - \int_k^{k+1} f(x) dx \\
&= \tfrac{1}{2}(f(k+1) + f(k)) - \int_k^{k+1} f(x) dx.
\end{aligned}$$

Summing up from $k = m$ to $n - 1$ yields the assertion of the theorem.

Using this theorem, we can construct an analytic continuation of the zeta function.

5.3. Theorem. *The Riemann zeta function can be analytically continued to a meromorphic function in the halfplane* $H(0) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 0\}$ *with a single pole of order 1 at* $s = 1$*. The continued function can be represented in* $H(0)$ *as*

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s\int_1^\infty \frac{\mathrm{saw}(x)}{x^{s+1}} dx.$$

Proof. Applying theorem 5.2 to the function $f(x) = 1/x^s$ we get

$$\sum_{n=1}^{N} \frac{1}{n^s} = \tfrac{1}{2}\Big(1 + \frac{1}{N^s}\Big) + \int_1^N \frac{dx}{x^s} - s\int_1^N \frac{\mathrm{saw}(x)}{x^{s+1}} dx.$$

For $\mathrm{Re}(s) > 1$ we have $\lim\limits_{N\to\infty} 1/N^s = 0$ and

$$\lim_{N\to\infty} \int_1^N \frac{dx}{x^s} = \lim_{N\to\infty} \frac{1}{1-s}\Big(\frac{1}{N^{s-1}} - 1\Big) = \frac{1}{s-1}.$$

Therefore we can pass to the limit $N \to \infty$ in the formula above and get for $\mathrm{Re}(s) > 1$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s\int_1^\infty \frac{\mathrm{saw}(x)}{x^{s+1}} dx. \tag{$*$}$$

We will now show that the integral

$$F(s) := \int_1^\infty \frac{\mathrm{saw}(x)}{x^{s+1}}\, dx$$

exists for all $s \in \mathbb{C}$ with $\sigma := \mathrm{Re}(s) > 0$ and represents a holomorphic function in the halfplane $H(0)$. This will then complete the proof of the theorem, since the right hand side of the formula $(*)$ defines a meromorphic continuation of the zeta function to $H(0)$ with a single pole at $s = 1$.

The existence of the integral follows from the estimate

$$\Big|\frac{\mathrm{saw}(x)}{x^{s+1}}\Big| \le \frac{1}{2} \cdot \frac{1}{x^{\sigma+1}},$$

since $\int_1^\infty (1/x^{\sigma+1})\, dx < \infty$ for $\sigma > 0$. To prove the holomorphy of F it suffices by the theorem of Morera to show that for all compact rectangles $R \subset H(0)$

$$\int_{\partial R} F(s) ds = 0.$$

This can be seen as follows: Since $\partial R \subset H(0)$ is compact, there exist a $\sigma_0 > 0$ such that $\mathrm{Re}(s) \ge \sigma_0$ for all $s \in \partial R$. Therefore we have on $\partial R \times [1, \infty[$ the majorization

$$\Big|\frac{\mathrm{saw}(x)}{x^{s+1}}\Big| \le \frac{1}{2} \cdot \frac{1}{x^{\sigma_0+1}}$$

and we can apply the theorem of Fubini

$$\begin{aligned}\int_{\partial R} F(s)\,ds &= \int_{\partial R}\int_1^\infty \frac{\operatorname{saw}(x)}{x^{s+1}}\,dx\,ds\\ &= \int_1^\infty \operatorname{saw}(x)\underbrace{\Big(\int_{\partial R}\frac{1}{x^{s+1}}\,ds\Big)}_{=0}\,dx = 0, \quad \text{q.e.d.}\end{aligned}$$

There exists also a proof of the holomorphy of F without recourse to Lebesgue integration theory: We write

$$F(s) = \int_1^\infty \frac{\operatorname{saw}(x)}{x^{s+1}}\,dx = \sum_{n=1}^\infty \int_n^{n+1}\frac{\operatorname{saw}(x)}{x^{s+1}}\,dx = \sum_{n=1}^\infty f_n(s)$$

with

$$f_n(s) = \int_n^{n+1}\frac{\operatorname{saw}(x)}{x^{s+1}}\,dx = \int_n^{n+1}\frac{x-n-\frac{1}{2}}{x^{s+1}}\,dx.$$

The function f_n is holomorphic in $\mathbb{C}$ (it is easily checked directly that $g(z) = \int_a^b t^z\,dt$ is holomorphic in the whole z-plane) and satisfies an estimate

$$|f_n(s)| \le \frac{1}{2n^{\sigma_0+1}} \quad \text{for all } s \in \overline{H(\sigma_0)}$$

Since $\sum_{n=1}^\infty 1/n^{\sigma_0+1} < \infty$ for all $\sigma_0 > 0$, the series $F = \sum_{n=1}^\infty f_n$ converges uniformly on every compact subset of $H(0)$. By a theorem of Weierstraß, the limit function F is holomorphic in $H(0)$.

5.4. Definition. The *Euler-Mascheroni* constant is defined as the limit

$$C := \lim_{N\to\infty}\Big(\sum_{n=1}^N \frac{1}{n} - \log N\Big).$$

The existence of this limit can be proved using the Euler-Maclaurin summation formula (5.2). This is left to the reader as an exercise.

5.5. Theorem. *There exist uniquely determined functions*

$$\beta_k : \mathbb{R} \longrightarrow \mathbb{R}, \quad k \in \mathbb{N}_1,$$

with the following properties:

i) All functions β_k are periodic with period 1, i.e. $\beta_k(x+n) = \beta_k(x)$ for all $n \in \mathbb{Z}$, and the functions β_k with $k \ge 2$ are continuous.

ii) $\beta_1 = \operatorname{saw}$.

iii) β_k is differentiable in $]0,1[$ and

$$\beta_k'(x) = \beta_{k-1}(x) \quad \text{for all } 0 < x < 1 \text{ and } k \geq 2.$$

iv) $\displaystyle\int_0^1 \beta_k(x)dx = 0$ for all $k \geq 1$

Proof. By condition iii), the function β_k is uniquely determined in the intervall $]0,1[$ by β_{k-1} up to an additive constant. This constant is uniquely determined by condition iv). Thus by ii)-iv), all β_k are uniquely determined in $]0,1[$, and by periodicity even in $\mathbb{R} \setminus \mathbb{Z}$. It remains to be shown that the definition of β_k, $k \geq 2$ can be extended continuously across the integer points. This is equivalent with

$$\lim_{\varepsilon \searrow 0} \beta_k(\varepsilon) = \lim_{\varepsilon \searrow 0} \beta_k(1-\varepsilon).$$

For $k \geq 2$ one has

$$\beta_k(1-\varepsilon) - \beta_k(\varepsilon) = \int_\varepsilon^{1-\varepsilon} \beta_{k-1}'(x)dx,$$

hence by iv)

$$\lim_{\varepsilon \searrow 0} (\beta_k(1-\varepsilon) - \beta_k(\varepsilon)) = \int_0^1 \beta_{k-1}'(x)dx = 0, \quad \text{q.e.d.}$$

Example. Let us calculate β_2. The condition

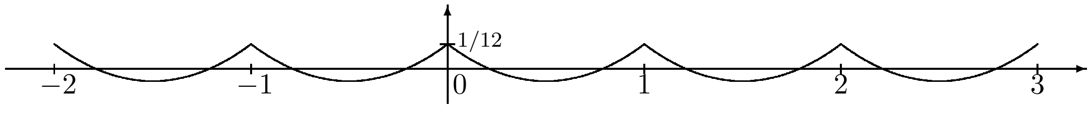
$$\beta_2'(x) = \beta_1(x) = x - \tfrac{1}{2} \quad \text{for } 0 < x < 1$$

leads to $\beta_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + c$ with an integration constant c. Since

$$\int_0^1 (\tfrac{1}{2}x^2 - \tfrac{1}{2}x)dx = \tfrac{1}{6} - \tfrac{1}{4} = -\tfrac{1}{12},$$

we have $c = \frac{1}{12}$, i.e.

$$\beta_2(x) = \tfrac{1}{2}x^2 - \tfrac{1}{2}x + \tfrac{1}{12} = \tfrac{1}{2}x(x-1) + \tfrac{1}{12} \quad \text{for } 0 \leq x \leq 1.$$



Graph of β_2

5.6. Theorem. *The functions β_n have the following Fourier expansions*

$$\beta_{2k}(x) = (-1)^{k-1} 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{(2\pi n)^{2k}}, \quad k \geq 1, \tag{1}$$

$$\beta_{2k+1}(x) = (-1)^{k-1} 2 \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{(2\pi n)^{2k+1}}, \quad k \geq 1, \tag{2}$$

which converge uniformly on $\mathbb{R}$.
Formula (2) is also valid for $k = 0$ and $x \in \mathbb{R} \smallsetminus \mathbb{Z}$.

Proof. a) We first calculate the Fourier series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$ of β_2. The coefficients c_n are given by the integral

$$c_n = \int_0^1 \beta_2(x) e^{-2\pi inx}\, dx.$$

By theorem 5.5.iv) we have $c_0 = 0$. Let now $n \neq 0$. Using partial integration we get

$$\int_0^1 xe^{-2\pi inx} dx = -\frac{1}{2\pi in} xe^{-2\pi inx}\Big|_0^1 + \frac{1}{2\pi in} \int_0^1 e^{-2\pi inx} dx = \frac{i}{2\pi n}$$

and

$$\int_0^1 x^2 e^{-2\pi inx} dx = -\frac{1}{2\pi in} x^2 e^{-2\pi inx}\Big|_0^1 + \frac{2}{2\pi in} \int_0^1 xe^{-2\pi inx} dx = \frac{i}{2\pi n} + \frac{2}{(2\pi n)^2},$$

hence

$$c_n = \int_0^1 \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x + \tfrac{1}{12}\right) e^{-2\pi inx} dx = \frac{1}{(2\pi n)^2}.$$

Thus we have the Fourier series

$$\beta_2(x) = \sum_{n\in\mathbb{Z}\smallsetminus 0} \frac{e^{2\pi in}}{(2\pi n)^2} = \sum_{n=1}^{\infty} \frac{e^{2\pi inx} + e^{-2\pi inx}}{(2\pi n)^2} = 2 \sum_{n=1}^{\infty} \frac{\cos(2\pi nx)}{(2\pi n)^2}.$$

By the general theory of Fourier series, the convergence is with respect to the L^2-norm $\|f\|_{L^2} = (\int_0^1 |f(x)|^2 dx)^{1/2}$, but since $\sum_{n=1}^{\infty} 1/n^2 < \infty$ and β_2 is continuous, we have even uniform convergence.

b) Since the right hand sides of the formulae of the theorem satisfy the same recursion and normalization relations (5.5.iii-iv) as the functions β_k, it follows that the given Fourier expansions are valid for all β_k, $k \geq 2$. To prove the formula for

$$\beta_1(x) = \operatorname{saw}(x) = -2 \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{2\pi n}, \quad x \in \mathbb{R} \smallsetminus \mathbb{Z},$$

it suffices to show that the series $\sum_{n=1}^{\infty}\frac{\sin(2\pi nx)}{2\pi n}$ converges uniformly on every interval $[\delta, 1-\delta]$, $0<\delta<\frac{1}{2}$, since then termwise differentiation of the Fourier series of β_2 is allowed. To simplify the notation we will prove the equivalent statement

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} \quad \textit{converges uniformly on } [\delta, 2\pi-\delta],\ (0<\delta<\pi).$$

Define

$$S_m(x) := \sum_{n=1}^{m}\sin nx = \operatorname{Im}\Big(\sum_{n=1}^{m} e^{inx}\Big).$$

For $\delta \le x \le 2\pi-\delta$ we have

$$|S_m(x)| \le \Big|\sum_{n=1}^{m} e^{inx}\Big| = \Big|\frac{e^{imx}-1}{e^{ix}-1}\Big| \le \frac{2}{|e^{ix/2}-e^{-ix/2}|} = \frac{1}{\sin\frac{x}{2}} \le \frac{1}{\sin\frac{\delta}{2}}.$$

It follows for $m \ge k > 0$

$$\begin{aligned}\Big|\sum_{n=k}^{m}\frac{\sin nx}{n}\Big| &= \Big|\sum_{n=k}^{m}\frac{S_n(x)-S_{n-1}(x)}{n}\Big| \\ &\le \Big|\sum_{n=k}^{m} S_n(x)\Big(\frac{1}{n}-\frac{1}{n+1}\Big) + \frac{S_m(x)}{m+1} - \frac{S_{k-1}(x)}{k}\Big| \\ &\le \frac{1}{\sin\frac{\delta}{2}}\Big(\frac{1}{k}-\frac{1}{m+1}+\frac{1}{m+1}+\frac{1}{k}\Big) \le \frac{2}{k\sin\frac{\delta}{2}},\end{aligned}$$

hence also

$$\Big|\sum_{n=k}^{\infty}\frac{\sin nx}{n}\Big| \le \frac{2}{k\sin\frac{\delta}{2}} \quad \text{for all } x\in[\delta, 2\pi-\delta],$$

which proves the asserted uniform convergence and thereby completes the proof of the theorem.

5.7. Definition. It follows immediately from (5.5.iii-iv) that β_n, restricted to the open interval $]0,1[$, is a polynomial of degree n with rational coefficients. The n-th *Bernoulli polynomial* $B_n(X)\in\mathbb{Q}[X]$ is defined by

$$\frac{B_n(x)}{n!} = \beta_n(x) \quad \text{for } 0<x<1,\ n\ge 1$$

and $B_0(X)=1$. The *Bernoulli numbers*[1] B_k are defined by

$$B_n := B_n(0), \quad n \ge 0.$$

[1]Strictly speaking, it is not correct to use the same symbol B_k for the Bernoulli polynomials and the Bernoulli numbers. However this notation is the usual one. To avoid confusion, we will always indicate the variable when we are dealing with Bernoulli polynomials.

We know already the first Bernoulli polynomials

$$B_1(X) = X - \tfrac{1}{2} \quad \text{and} \quad B_2(X) = X(X-1) + \tfrac{1}{6},$$

hence $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$.

An easy consequence of theorem 5.6 is

5.8. Theorem. *For the Bernoulli numbers the following relations hold:*

i) $B_{2k+1} = 0$ for all $k \geq 1$.

ii) $B_{2k} = (-1)^{k-1} \dfrac{2(2k)!}{(2\pi)^{2k}} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, hence

$$\zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!} |B_{2k}| \quad \text{for all } k \geq 1.$$

iii) $\operatorname{sign}(B_{2k}) = (-1)^{k-1}$ for all $k \geq 1$.

Remarks. a) Formula ii) of the theorem says in particular

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

which was already used in the previous chapter.

b) Since $\lim\limits_{\sigma\to\infty} \zeta(\sigma) = 1$, formula ii) shows the asymptotic growth of the Bernoulli numbers B_{2k}

$$|B_{2k}| \sim \frac{2(2k)!}{(2\pi)^{2k}} \quad \text{for } k \to \infty.$$

5.9. Theorem (Generating function for the Bernoulli polynomials). *For fixed $x \in \mathbb{R}$, the function $\dfrac{te^{xt}}{e^t - 1}$ is a complex analytic function of t with a removable singularity at $t = 0$. The Taylor expansion at $t = 0$ of this function has the form*

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

In particular, for $x = 0$ one has

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

Proof. Define $B_n(x)$ by the above Taylor expansions. We will show that

(a) $B_0(x) = 1, \;\; B_1(x) = x - \frac{1}{2}$,

(b) $B_n'(x) = nB_{n-1}(x), \;\; (n \geq 1)$,

(c) $\int_0^1 B_n(x)dx = 0, \;\; (n \geq 1)$.

Then theorem 5.5 implies $\dfrac{B_n(x)}{n!} = \beta_n(x)$ for $0 < x < 1$ and all $n \geq 1$.

Proof of (a)

$$\begin{aligned}
\frac{te^{xt}}{e^t - 1} &= \frac{t(1 + xt + O(t^2))}{t + \frac{1}{2}t^2 + O(t^3)} = \frac{1 + xt + O(t^2)}{1 + \frac{1}{2}t + O(t^2)} \\
&= (1 + xt)(1 - \tfrac{1}{2}t) + O(t^2) = 1 + (x - \tfrac{1}{2})t + O(t^2),
\end{aligned}$$

which shows $B_0(x) = 1$ and $B_1(x) = x - \frac{1}{2}$.

Proof of (b) We calculate $\dfrac{\partial}{\partial x}\dfrac{te^{xt}}{e^t - 1}$ in two ways

$$\frac{\partial}{\partial x}\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n'(x)}{n!} t^n$$

and

$$\frac{\partial}{\partial x}\frac{te^{xt}}{e^t - 1} = \frac{t^2 e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^{n+1} = \sum_{n=1}^{\infty} \frac{B_{n-1}(x)}{(n-1)!} t^n$$

Comparing coefficients we get $B_n'(x) = nB_{n-1}(x)$.

Proof of (c)

$$\int_0^1 \frac{te^{xt}}{e^t - 1}\, dx = \frac{e^{xt}}{e^t - 1}\bigg|_{x=0}^{x=1} = \frac{e^t}{e^t - 1} - \frac{1}{e^t - 1} = 1.$$

On the other hand

$$\int_0^1 \frac{te^{xt}}{e^t - 1}\, dx = \sum_{n=1}^{\infty} \left(\int_0^1 B_n(x)dx \right) \frac{t^n}{n!}.$$

Comparing coefficients, we get $\int_0^1 B_n(x)dx = 0$ for all $n \geq 1$, q.e.d.

5.10. Recursion formula. Theorem 5.9 can be used to derive a recursion formula for the Bernoulli numbers. Since $(e^t - 1)/t = \sum\limits_{n=1}^{\infty} t^{n-1}/n!$, we have

$$\left(\sum_{k=0}^{\infty} \frac{B_k}{k!} t^k \right) \left(\sum_{\ell=0}^{\infty} \frac{1}{(\ell + 1)!} t^\ell \right) = 1.$$

The Cauchy product $\sum_{n=0}^{\infty} c_n t^n$ of the two series has coefficients

$$c_n = \sum_{k=1}^{n} \frac{B_k}{k!(n-k+1)!} = \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} B_k.$$

Hence comparing coefficients we get $B_0 = 1$ and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad \text{for all } n \geq 1.$$

With this formula one can recursively calculate all B_n. The first non zero coefficients are

k	0	1	2	4	6	8	10	12	14	16
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$

5.11. Theorem (Euler-Maclaurin II). *Let x_0 be a real number and $f : [x_0, \infty[\to \mathbb{C}$ a $2r$-times continuously differentiable function. Then we have for all integers $n \geq m \geq x_0$ and all $r \geq 1$*

$$\begin{aligned}\sum_{k=m}^{n} f(k) &= \tfrac{1}{2}(f(m) + f(n)) + \int_m^n f(x)dx \\ &\quad + \sum_{k=1}^{r} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) - \int_m^n \frac{\widetilde{B}_{2r}(x)}{(2r)!} f^{(2r)}(x)dx\end{aligned}$$

Here $\widetilde{B}_{2r}(x)$ is the periodic function defined by $\widetilde{B}_{2r}(x) := B_{2r}(x - \lfloor x \rfloor) = (2r)!\beta_{2r}(x)$.

Proof. We start with theorem 5.2

$$\sum_{k=m}^{n} f(k) = \tfrac{1}{2}(f(m) + f(n)) + \int_m^n f(x)dx + \int_m^n \operatorname{saw}(x) f'(x)dx.$$

and evaluate the last integral by partial integration.

Since $\beta_2'(x) = \operatorname{saw}(x)$ for $k < x < k+1$ and β_2 is continuous and periodic, we get

$$\begin{aligned}\int_m^n \operatorname{saw}(x) f'(x)dx &= \sum_{k=m}^{n-1} \int_k^{k+1} \operatorname{saw}(x) f'(x)dx \\ &= \sum_{k=m}^{n-1} \beta_2(x) f'(x)\Big|_k^{k+1} - \sum_{k=m}^{n-1} \int_k^{k+1} \beta_2(x) f''(x)dx \\ &= \sum_{k=m}^{n-1} \left(\beta_2(k+1) f'(k+1) - \beta_2(k) f'(k)\right) - \int_m^n \beta_2(x) f''(x)dx \\ &= \frac{B_2}{2!}(f'(n) - f'(m)) - \int_m^n \beta_2(x) f''(x)dx.\end{aligned}$$

This proves the case $r = 1$ of the theorem. The general case is proved by induction.
Induction step $r \to r+1$.

$$\begin{aligned}
-\int_m^n \beta_{2r}(x) f^{(2r)}(x)dx &= -\beta_{2r+1}(x) f^{(2r)}(x)\Big|_m^n + \int_m^n \beta_{2r+1}(x) f^{(2r+1)}(x)dx \\
&= \int_m^n \beta_{2r+1}(x) f^{(2r+1)}(x)dx \qquad [\text{since } \beta_{2r+1}(k) = \frac{B_{2r+1}}{(2r+1)!} = 0] \\
&= \beta_{2r+2}(x) f^{(2r+1)}(x)\Big|_m^n - \int_m^n \beta_{2r+2}(x) f^{(2r+2)}(x)dx \\
&= \frac{B_{2r+2}}{(2r+2)!}(f^{(2r+1)}(n) - f^{(2r+1)}(m)) - \int_m^n \beta_{2r+2}(x) f^{(2r+2)}(x)dx.
\end{aligned}$$

This proves the assertion for $r+1$.

Remark. If f is infinitely often differentiable and we pass to the limit $r \to \infty$, the "error term"

$$\int_m^n \frac{\widetilde{B}_{2r}(x)}{(2r)!} f^{(2r)}(x)dx$$

will in general not converge to 0. In case f is real and $f^{(2r)}$ does not change sign in the interval $[m, n]$, one has the following estimate

$$\Big|\int_m^n \frac{\widetilde{B}_{2r}(x)}{(2r)!} f^{(2r)}(x)dx\Big| \le \frac{|B_{2r}|}{(2r)!}\Big|\int_m^n f^{(2r)}(x)dx\Big| = \frac{|B_{2r}|}{(2r)!}|f^{(2r-1)}(n) - f^{(2r-1)}(m)|,$$

which means that the error of the approximation

$$\sum_{k=m}^n f(k) \approx \tfrac{1}{2}(f(m) + f(n)) + \int_m^n f(x)dx + \sum_{k=1}^r \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right)$$

is by absolute value not larger than the last term of the sum. Hence by increasing r one gets better approximations as long as the absolute values of the added terms decrease.

5.12. Theorem. *The Riemann zeta function can be analytically continued to a meromorphic function in the whole plane $\mathbb{C}$ with a single pole of order 1 at $s = 1$. For* $\mathrm{Re}(s) > 1 - 2r$, *the continued function can be represented as*

$$\begin{aligned}
\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + \sum_{k=1}^r \frac{B_{2k}}{(2k)!}\, s(s+1)\cdot\ldots\cdot(s+2k-2) \\
- s(s+1)\cdot\ldots\cdot(s+2r-1)\int_1^\infty \frac{\widetilde{B}_{2r}(x)}{(2r)!}\cdot\frac{1}{x^{s+2r}}\,dx.
\end{aligned}$$

Proof. This is proved by applying theorem 5.11 to the sum $\sum_{k=1}^n 1/k^s$ and passing to the limit $n \to \infty$. That the last integral defines a holomorphic function for $\mathrm{Re}(s) > 1 - 2r$, follows from the fact that the function $\widetilde{B}_{2r}(x)$ is bounded and

$$\Big|\frac{1}{x^{s+2r}}\Big| \le \frac{1}{x^{1+\delta}} \quad \text{for all } s \in \mathbb{C} \text{ with } \mathrm{Re}(s) \ge 1 - 2r + \delta.$$