4. Riemann Zeta Function. Euler Product

4.1. Definition. For a complex $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Riemann zeta function is defined by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Let us first study the convergence of this infinite series. Following an old tradition, we denote the real and imaginary part of s by σ resp. t, i.e.

$$s = \sigma + it, \quad \sigma, t \in \mathbb{R}.$$

We have

$$\frac{1}{n^s} = n^{-s} = e^{-s\log n} = e^{-\sigma\log(n) - it\log n} = \frac{1}{n^\sigma} e^{-it\log n},$$

therefore

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^\sigma}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ converges for all real $\sigma > 1$, we see that the zeta series converges absolutely and uniformly in every halfplane $\overline{H(\sigma_0)}$, $\sigma_0 > 1$, where

$$H(\sigma_0) := \{s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_0\}.$$

It follows by a theorem of Weierstrass that ζ is a holomorphic (= regular analytic) function in the halfplane

$$H(1) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}.$$

We will see later that ζ can be continued analytically to a meromorphic function in the whole complex plane $\mathbb{C}$, which is holomorphic in $\mathbb{C} \smallsetminus \{1\}$ and has a pole of first order at $s = 1$. A weaker statement is

4.2. Proposition. $\lim_{\sigma \searrow 1} \zeta(\sigma) = \infty.$

Proof. Let $R > 0$ be any given bound. Since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, there exists an $N > 1$ such that

$$\sum_{n=1}^{N} \frac{1}{n} \geq R + 1.$$

The function $\sigma \mapsto \sum_{n=1}^{N} \frac{1}{n^\sigma}$ is continuous on $\mathbb{R}$, hence there exists an $\varepsilon > 0$ such that

$$\sum_{n=1}^{N} \frac{1}{n^\sigma} \geq R \quad \text{for all } \sigma \text{ with } \sigma < 1 + \varepsilon.$$

A fortiori we have $\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \geq R$ for all $1 < \sigma < 1 + \varepsilon$. This proves the proposition.

4.3. Theorem (Euler product). *For all $s \in \mathbb{C}$ with* $\operatorname{Re}(s) > 1$ *one has*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}},$$

where the product is extended over the set $\mathbb{P}$ of all primes.

Proof. Since $|p^{-s}| < 1/p \leq 1/2$, we can use the geometric series

$$\frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} \frac{1}{p^{ks}},$$

which converges absolutely. If $\mathcal{P} \subset \mathbb{P}$ is any finite set of primes, the product

$$\prod_{p \in \mathcal{P}} \Big(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \Big) = \prod_{p \in \mathcal{P}} \Big(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots \Big)$$

can be calculated by termwise multiplication and we obtain

$$\prod_{p \in \mathcal{P}} \Big(\sum_{k=0}^{\infty} \frac{1}{p^{ks}} \Big) = \sum_{n \in \mathbb{N}(\mathcal{P})} \frac{1}{n^s},$$

where $\mathbb{N}(\mathcal{P})$ is the set of all positive integers n whose prime decomposition contains only primes from the set $\mathcal{P}$. (Here the unique prime factorization is used.) Letting $\mathcal{P} = \mathcal{P}_m$ be set of all primes $\leq m$ and passing to the limit $m \to \infty$, we obtain the assertion of the theorem.

Remark. The Euler product can be used to give another proof of the infinitude of primes. If the set $\mathbb{P}$ of all primes were finite, the Euler product $\prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}$ would be continuous at $s = 1$, which contradicts the fact that $\lim_{\sigma \searrow 1} \zeta(\sigma) = \infty$.

4.4. We recall some facts from the theory of analytic functions of a complex variable about infinite products. Let $G \subset \mathbb{C}$ be an open set. For a continuous function $f : G \to \mathbb{C}$ and a compact subset $K \subset G$ we define the maximum norm

$$\|f\|_K := \sup\{|f(z)| : z \in K\} \in \mathbb{R}_+.$$

(The supremum is $< \infty$ since f is continous.) Let now $f_\nu : G \to \mathbb{C}$, $\nu \geq 1$, be a sequence of holomorphic functions. The infinite product

$$F(z) := \prod_{\nu=1}^{\infty}(1 + f_\nu(z))$$

is said to be *normally convergent* on a compact subset $K \subset G$, if

$$\sum_{\nu=1}^{\infty} \|f_\nu\|_K < \infty.$$

In this case, the product converges absolutely and uniformly on K. (The converse is not true, as can be seen by taking the constant functions $f_\nu = -\frac{1}{2}$ for all ν.) The product is said to be normally convergent in G if it converges normally on any compact subset of $K \subset G$. The limit F of a normally convergent infinite product of holomorphic functions $1 + f_\nu$ is again holomorphic and $F(z_0) = 0$ for a particular point $z_0 \in G$ if and only if one of the factors vanishes in z_0.

4.5. Theorem. *The Riemann zeta function has no zeroes in the half plane*

$$H(1) = \{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}.$$

For its inverse one has

$$\frac{1}{\zeta(s)} = \prod_{p \in \mathbb{P}}\Big(1 - \frac{1}{p^s}\Big) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where μ is the Möbius function.

Proof. The first assertion follows from the fact that the Euler product for the zeta function converges normally in $H(1)$ and all factors $(1-p^{-s})^{-1}$ have no zeroes in $H(1)$. Inverting the product representation for $1/\zeta(s)$ yields $1/\zeta(s) = \prod(1 - p^{-s})$. To prove the last equation, let $\mathcal{P}$ a finite set of primes and $\mathbb{N}'(\mathcal{P})$ the set of all positive integers n that can be written as a product $n = p_1 p_2 \cdot \ldots \cdot p_r$ of distinct primes $p_j \in \mathcal{P}$, $(r \geq 0)$. Then, since $(-1)^r = \mu(p_1 \cdot \ldots \cdot p_r)$,

$$\prod_{p \in \mathcal{P}}\Big(1 - \frac{1}{p^s}\Big) = \sum_{n \in \mathbb{N}'(\mathcal{P})} \frac{\mu(n)}{n^s}.$$

Letting $\mathcal{P} = \mathcal{P}_m$ be set of all primes $\leq m$ and passing to the limit $m \to \infty$, we obtain the assertion of the theorem. Note that $\mu(n) = 0$ for all $n \in \mathbb{N}_1 \smallsetminus \bigcup_m \mathbb{N}'(\mathcal{P}_m)$.

4.6. We recall now some facts about the logarithm function. (By logarithm we always mean the natural logarithm with basis $e = 2.718\ldots$.) We have the Taylor expansion

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} \quad \text{for all } z \in \mathbb{C} \text{ with } |z| < 1.$$

From this follows

$$\log\Big(\frac{1}{1-z}\Big) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for all } z \in \mathbb{C} \text{ with } |z| < 1.$$

(Of course here the principal branch of the logarithm with $\log(1) = 0$ is understood.)

If $f : G \to \mathbb{C}$ is a holomorphic function without zeroes in a simply connected domain $G \subset \mathbb{C}$, then there exists a holomorphic branch of the logarithm of f, i.e. a holomorphic function

$$\log f : G \to \mathbb{C} \quad \text{with} \quad e^{(\log f)(z)} = f(z) \text{ for all } z \in G.$$

This function $\log f$ is uniquely determined up to an additive constant $2\pi i n$, $n \in \mathbb{Z}$.

Since the zeta function has no zeroes in the simply connected halfplane $H(1)$, we can form the logarithm of the zeta function, where we select the branch of $\log \zeta$ that takes real values on the real half line $]1, \infty[$.

4.7. Theorem. *For the logarithm of the zeta function in the halfplane $H(1)$, the following equation holds:*

$$\log \zeta(s) = \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p \in \mathbb{P}} \frac{1}{p^{ks}}.$$

The function

$$F(s) := \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p \in \mathbb{P}} \frac{1}{p^{ks}}$$

is bounded in $H(1)$.

Remark. If one defines the *prime zeta function* by

$$P(s) := \sum_{p \in \mathbb{P}} \frac{1}{p^s} \quad \text{for } s \in H(1),$$

the formula of the theorem may be written as

$$\log \zeta(s) = \sum_{k=1}^{\infty} \frac{P(ks)}{k} = P(s) + F(s), \quad \text{where} \quad F(s) = \sum_{k=2}^{\infty} \frac{P(ks)}{k}.$$

Proof. Using the Euler product we obtain

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathbb{P}} \log\Big(\frac{1}{1-p^{-s}}\Big) = \sum_{p \in \mathbb{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} = \sum_{k=1}^{\infty} \sum_{p \in \mathbb{P}} \frac{1}{kp^{ks}} \\ &= \sum_{p \in \mathbb{P}} \frac{1}{p^s} + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p \in \mathbb{P}} \frac{1}{p^{ks}}. \end{aligned}$$

To prove the boundedness of

$$F(s) = \sum_{k=2}^{\infty} \frac{1}{k} \sum_{p \in \mathbb{P}} \frac{1}{p^{ks}} = \sum_{k=2}^{\infty} \frac{P(ks)}{k}$$

in $H(1)$, we use the estimate (with $\sigma = \mathrm{Re}(s) > 1$)

$$\begin{aligned} |P(ks)| &\le P(k\sigma) \le P(k) = \sum_{p \in \mathbb{P}} \frac{1}{p^k} \le \sum_{n=2}^{\infty} \frac{1}{n^k} \\ &\le \sum_{n=2}^{\infty} \int_{n-1}^{n} \frac{dx}{x^k} = \int_1^{\infty} \frac{dx}{x^k} = \frac{1}{k-1} \end{aligned}$$

and obtain for all $s \in H(1)$

$$|F(s)| \le \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = 1, \quad \text{q.e.d.}$$

4.8. Corollary (Euler).

$$\sum_{p \in \mathbb{P}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots = \infty.$$

Proof. Since the difference $|P(s) - \log \zeta(s)|$ is bounded for $\mathrm{Re}(s) > 1$ we get, using proposition 4.2,

$$\lim_{\sigma \searrow 1} P(\sigma) = \lim_{\sigma \searrow 1} \Big(\sum_{p \in \mathbb{P}} \frac{1}{p^{\sigma}} \Big) = \infty.$$

This implies the assertion.

Remark. The corollary gives another proof that there are infinitely many primes, but says more. Comparing with

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

we can conclude that the density of primes is in some sense greater than the density of square numbers.

The following theorem is a variant of theorem 4.7 and gives an interesting formula for the difference between $P(s)$ and $\log \zeta(s)$.

4.9. Theorem. *We have the following representation of the prime zeta function for* $\mathrm{Re}(s) > 1$

$$P(s) = \sum_{p \in \mathbb{P}} \frac{1}{p^s} = \log \zeta(s) + \sum_{k=2}^{\infty} \frac{\mu(k)}{k} \log \zeta(ks).$$

Proof. We start from the formula of theorem 4.7

$$\log \zeta(s) = \sum_{k=1}^{\infty} \frac{P(ks)}{k}.$$

We have as in the proof of theorem 4.7 the estimate

$$|P(ks)| \le P(k\sigma) \le \frac{1}{k\sigma - 1} \le \frac{2}{k\sigma}, \qquad \text{(where } \sigma = \mathrm{Re}(s)),$$

which implies

$$|\log \zeta(s)| \le \sum_{k=1}^{\infty} \frac{2}{k^2\sigma} = \frac{2}{\sigma}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{2\zeta(2)}{\sigma} =: \frac{c}{\sigma}$$

with the constant $c = 2\zeta(2)$. Therefore the series $\sum_{k=1}^{\infty}(\mu(k)/k)\log\zeta(ks)$ converges absolutely:

$$\sum_{k=1}^{\infty}\left|\frac{\mu(k)}{k}\log\zeta(ks)\right| \le \sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{c}{k\sigma} = \frac{c\zeta(2)}{\sigma} < \infty.$$

Substituting $\log\zeta(ks) = \sum_{\ell=1}^{\infty} P(k\ell s)/\ell$ we get

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\mu(k)}{k}\log\zeta(ks) &= \sum_{k,\ell=1}^{\infty}\frac{\mu(k)P(k\ell s)}{k\ell} = \sum_{n=1}^{\infty}\sum_{k\ell=n}\mu(k)\frac{P(k\ell s)}{k\ell}\\ &= \sum_{n=1}^{\infty}\sum_{k|n}\mu(k)\frac{P(ns)}{n} = \sum_{n=1}^{\infty}\delta_1(n)\frac{P(ns)}{n}\\ &= P(s), \qquad \text{q.e.d.}\end{aligned}$$

We conclude this chapter with an interesting application of therem 4.5.

4.10. Theorem. *The probability that two random numbers $m, n \in \mathbb{N}_1$ are coprime is $6/\pi^2 \approx 61\%$, more precisely: For real $x \ge 1$ let*

$$\mathrm{Copr}(x) := \{(m,n) \in \mathbb{N}_1 \times \mathbb{N}_1 : m, n \le x \text{ and } m, n \text{ coprime}\}.$$

Then

$$\lim_{x\to\infty}\frac{\#\mathrm{Copr}(x)}{x^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}.$$

Proof. Let $A(x)$ be the set of all pairs m, n of integers with $1 \le m, n \le x$ and

$$A_k(x) := \{(n,m) \in A(x) : \gcd(m,n) = k\}.$$

Then $A(x)$ is the disjoint union of all $A_k(x)$, $k = 1, 2, \ldots, \lfloor x \rfloor$, and for every k we have a bijection

$$\mathrm{Copr}\Big(\frac{x}{k}\Big) \longrightarrow A_k(x), \quad (m,n) \mapsto (km, kn).$$

Therefore

$$\sum_{k \leqslant x} \#\mathrm{Copr}\Big(\frac{x}{k}\Big) = \lfloor x \rfloor^2.$$

Now we can apply the inversion formula of theorem 3.16 and obtain

$$\#\mathrm{Copr}(x) = \sum_{k \leqslant x} \mu(k) \Big\lfloor \frac{x}{k} \Big\rfloor^2.$$

Since $0 \le (x/k) - \lfloor x/k \rfloor < 1$, it follows that $(x/k)^2 - \lfloor x/k \rfloor^2 < 2x/k$, hence

$$\Big| \#\mathrm{Copr}(x) - \sum_{k \leqslant x} \mu(k) \Big(\frac{x}{k}\Big)^2 \Big| \le 2x \sum_{k \leqslant x} \frac{1}{k} \le 2x(1 + \log x) = O(x \log x),$$

so we can write

$$\frac{\#\mathrm{Copr}(x)}{x^2} = \sum_{k \leqslant x} \frac{\mu(k)}{k^2} + O\Big(\frac{\log x}{x}\Big).$$

On the other hand $\sum_{k=1}^{\infty} \mu(k)/k^2 = 1/\zeta(2)$ by theorem 4.5, hence

$$\Big| \sum_{k \leqslant x} \frac{\mu(k)}{k^2} - \frac{1}{\zeta(2)} \Big| \le \sum_{k > x} \frac{1}{k^2} = O\Big(\frac{1}{x}\Big).$$

Combining this with the previous estimate yields

$$\frac{\#\mathrm{Copr}(x)}{x^2} = \frac{1}{\zeta(2)} + O\Big(\frac{\log x}{x}\Big),$$

which implies the assertion of the theorem.

Remark. The fact $\zeta(2) = \dfrac{\pi^2}{6}$ will be proven in the next chapter.