Complete intersections in affine algebraic varieties and Stein spaces [1]

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Introduction

Let $(X, \mathcal{O}_X)$ be an affine algebraic variety (or an affine scheme, or a Stein space) and $Y \subset X$ a Zariski-closed (resp. analytic) subspace. We want to describe Y set-theoretically (or ideal-theoretically) by global functions, i.e. find elements $f_1, \ldots, f_N \in \Gamma(X, \mathcal{O}_X)$ such that

$$Y = \{x \in X : f_1(x) = \ldots = f_N(x) = 0\},$$

resp. such that $f_1, \ldots, f_N$ generate the ideal of Y (which is a stronger condition). The problem we consider here is how small the number N can be chosen. If in particular N can be chosen equal to the codimension of Y, then Y is called a set-theoretic (resp. ideal-theoretic) complete intersection.
In these lectures we discuss some results with respect to this problem in the algebraic and analytic case. In considering these cases simultaneously, it is interesting to note the analogies and differences of the methods and results. For this purpose we adopt also a more geometric point of view for the algebraic case. We hope that some proofs become more intuitive in this way.

I. Estimation of the number of equations necessary to describe an algebraic (resp. analytic) set

We begin with the following classical result on the set-theoretic description of an algebraic set.

1.1. Theorem (Kronecker 1882). *Let $(X, \mathcal{O}_X)$ be an n-dimensional affine, algebraic space (or the affine scheme of an n-dimensional noetherian ring) and $Y \subset X$ an algebraic subset. Then there exist functions $f_1, \ldots, f_{n+1} \in \Gamma(X, \mathcal{O}_X)$ such that*

$$Y = V(f_1, \ldots, f_{n+1}) := \{x \in X : f_1(x) = \ldots = f_{n+1}(x) = 0\}.$$

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Proof (due to Van der Waerden 1941). We prove by induction the following statement

$(A.k)$ There exist $f_1, \dots, f_k \in \Gamma(X, \mathcal{O}_X)$ such that

$$V(f_1, \dots, f_k) = Y \cup Z_k,$$

where Z_k is an algebraic subset of X with $\operatorname{codim} Z_k \geqslant k$.

The statement $(A.0)$ is trivial, whereas $(A.n{+}1)$ gives the theorem. So it remains to prove the induction step

$(A.k) \to (A.k{+}1)$. Let

$$Z_k = Z_k^1 \cup \ldots \cup Z_k^s$$

be the decomposition of Z_{k_i} into its irreducible components. We may suppose that none of the Z_k^i is contained in Y. Choose a point $p_i \in Z_k^i \smallsetminus Y$ for $i = 1, \dots, s$. Now it is easy to construct a function $f_{k+1} \in \Gamma(X, \mathcal{O}_X)$ with

$$f_{k+1}|_Y = 0 \quad \text{and} \quad f_{k+1}(p_i) \neq 0 \quad \text{for} \quad i = 1, \dots, s.$$

Then $V(f_1, \dots, f_{k+1}) = Y \cup Z_{k+1}$ with

$$\operatorname{codim} Z_{k+1} > \operatorname{codim} Z_k \geqslant k.$$

□

We want to give an example which shows that in general n equations do not suffice.

1.2. Example. Let $\overline{X}$ be an elliptic curve over $\mathbb{C}$, considered as a torus

$$\overline{X} = \mathbb{C}/\Gamma, \quad \Gamma \text{ lattice.}$$

Let $p \in \overline{X}$ be an arbitrary point. Then

$$X := \overline{X} \smallsetminus \{p\}$$

is a 1-dimensional affine algebraic variety. Let $Y := \{q\}$ with some $q \in X$. Let P, $Q \in \mathbb{C}$ be representatives of p and q respectively.

Claim. If $P - Q \notin \mathbb{Q} \cdot \Gamma$, then there exists no function $f \in \Gamma(X, \mathcal{O}_{X_{alg}})$ such that

$$Y = \{q\} = V(f).$$

Proof. Such a function f can be considered as a meromorphic function on $\overline{X}$, with poles only in p and zeros only in q. Let $k > 0$ be the vanishing order of f at q. Then k is also the order of the pole of f in p. Thus $k \cdot q - k \cdot p$ would be a principal divisor on $\overline{X}$. By the theorem of Abel, this implies

$$kQ - kP \in \Gamma.$$

But this contradicts our assumption $P - Q \notin \mathbb{Q} \cdot \Gamma$. Hence f cannot exist. $\square$

Remark. If we work in the analytic category, i.e. consider X as an open Riemann surface, then there exists a holomorphic function $f \in \Gamma(X, \mathcal{O}_X)$ which vanishes precisely in q of order one. This is a special case of the theorem of Weierstraß for open Riemann surfaces, proved by Behnke/Stein 1948, that every divisor on an open Riemann surface is the divisor of a meromorphic function (see e.g. [11]).

Open Riemann surfaces are special cases of Stein spaces, which are the analogue of affine algebraic varieties in complex analysis. A complex space $(X, \mathcal{O}_X)$ is called a Stein space, if the following conditions are satisfied:

i) X is *holomorphically separable*, i.e. given two points $x \neq y$ on X, there exists a holomorphic function $f \in \Gamma(X, \mathcal{O}_X)$ such that $f(x) \neq f(y)$.

ii) X is *holomorphically convex*, i.e. given a sequence $x_1, x_2, \ldots$ of points on X without point of accumulation, there exists $f \in \Gamma(X, \mathcal{O}_X)$ with $\limsup_{k\to\infty} |f(x_k)| = \infty$.

For the general theory of Stein spaces we refer to [18].

In an n-dimensional Stein space, n equations always suffice to describe an analytic subset:

1.3. Theorem (Forster/Ramspott [12]**).** *Let X be an n-dimensional Stein space and $Y \subset X$ a (closed) analytic subset. Then there exist n holomorphic functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

$$Y = V(f_1, \ldots, f_n).$$

Proof. We prove the theorem by induction on n. In order to do so, we have to prove a more precise version, namely, given a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ with $V(\mathcal{I}) = Y$, we can find functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{I})$ such that $Y = V(f_1, \ldots, f_n)$.

$n = 1$. This is a little generalization of the Weierstraß theorem for open Riemann surfaces. It follows from the fact that for 1-dimensional Stein spaces (which may have singularities) one has $H^1(X, \mathcal{O}_X^*) = H^2(X, \mathbb{Z}) = 0$.

$n-1 \to n$. First one can find a function $f \in \Gamma(X, \mathcal{I})$ such that

$$V(f) = Y \cup Z, \text{ where } \dim Z \leqslant n-1.$$

Let $\mathcal{J} \subset \mathcal{O}_Z$ be the image of $\mathcal{I}$ under the restriction morphism $\mathcal{O}_X \to \mathcal{O}_Z$. Then $\mathcal{J}$ is a coherent ideal sheaf with $V_Z(\mathcal{J}) = Z \cap Y$, and we can apply the induction hypothesis to find $g_1, \ldots, g_{n-1} \in \Gamma(Z, \mathcal{J})$ such that

$$Z \cap Y = V_Z(g_1, \ldots, g_{n-1}).$$

Since X is Stein, the morphism $\Gamma(X, \mathcal{I}) \to \Gamma(Z, \mathcal{J})$ is surjective. Let $f_1, \ldots, f_{n-1} \in \Gamma(X, \mathcal{I})$ be functions that are mapped onto $g_1, \ldots, g_{n-1}$, then

$$Y = V_X(f_1, \ldots, f_{n-1}, f).$$

$\square$

As we have seen, in the algebraic case n equations do not suffice in general. However, if one can factor out an affine line from the affine algebraic variety, n equations will suffice.

1.4. Theorem (Storch [33]**, Eisenbud/Evans** [7]**).** *Let X be an affine algebraic space of the form $X = X_1 \times \mathbb{A}^1$, where X_1 is an affine algebraic space of dimension $n-1$ (or more generally $X = \operatorname{Spec} R[T]$, where R is an $(n-1)$-dimensional noetherian ring). Then for every algebraic subset $Y \subset X$ there exist n functions $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ such that*

$$Y = V(f_1, \ldots, f_n).$$

In order to carry out the proof, we need a sharper version: Let there be given an ideal $\mathfrak{a} \in \Gamma(X, \mathcal{O}_X)$ such that $V(\mathfrak{a}) = Y$. Then the functions $f_1, \ldots, f_n$ can be chosen in $\mathfrak{a}$.
However, by the Hilbert Nullstellensatz the rough version of the theorem implies the sharper version.

Proof by induction on n. We may suppose X to be reduced.

$n = 1$. Then X_1 is a finite set of points, so X is a finite union of affine lines and the assertion is trivial.

Induction step $n-1 \to n$. We have

$$\Gamma(X, \mathcal{O}_X) = R[T], \quad \text{where} \quad R = \Gamma(X_1, \mathcal{O}_{X_1}).$$

Let S be the set of non-zero divisors of R and

$$K = Q(R) = S^{-1}R$$

the total quotient ring of R. We have

$$K = K_1 \times \ldots \times K_r$$

where every K_j is a field. Let $\tilde{\mathfrak{a}} = \mathfrak{a}K[T]$. Since $K[T]$ is a principal ideal ring, there is an $f \in R[T]$ such that $\tilde{\mathfrak{a}} = K[T]f$. Let $\mathfrak{b} = R[T]f$. Then there exists a certain $s \in S$ such that

$$(*) \quad \mathfrak{a} \supset \mathfrak{b} \supset s\mathfrak{a}.$$

Let $X_2 := V_{X_1}(s)$. Then $(*)$ implies

$$Y \subset V(f) \subset Y \cup (X_2 \times \mathbb{A}^1).$$

We have $\dim X_2 \leqslant n - 2$. Applying the induction hypothesis to $X_2 \times \mathbb{A}^1$, the algebraic subset $(X_2 \times \mathbb{A}^1) \cap Y$ and the ideal $\mathfrak{a}_2 := \mathrm{Im}(\mathfrak{a} \to (R/s)[T])$, we get the theorem. $\square$

1.5. Corollary. *In affine n-space $\mathbb{A}^n$, every algebraic subset is the set of zeros of n polynomials.*

Remark. Also in projective n-space $\mathbb{P}^n$ every algebraic set can be described (set-theoretically) by n homogeneous polynomials. This can be proved by methods similar to the affine case, cf. Eisenbud/Evans [7]. For $n = 3$ this had been already proved by Kneser [21].

To conclude this section, we formulate the following

Problem. Find a smooth n-dimensional affine algebraic variety X and a hypersurface $Y \subset X$ that cannot be described set-theoretically by less than $n+1$ functions.

Example 1.2 is the case $n = 1$. In higher dimensions the problem appears to be much more difficult.

II. Estimation of the number of elements necessary to generate a module over a noetherian ring

Let R be a noetherian ring and M a finitely generated R-module. We want to estimate the minimal number of generators of M over R by a local-global principle. For this purpose, we associate to R the maximal ideal space

$$X = \mathrm{Specm}(R),$$

endowed with the Zariski topology. Localization of R gives us a sheaf of rings $\mathcal{O}_X$ on X such that

$$R = \Gamma(X, \mathcal{O}_X).$$

To the R-module M there is associated a coherent $\mathcal{O}_X$-module sheaf $\mathcal{M}$ such that

$$M = \Gamma(X, \mathcal{M}).$$

We use the well-known fact: A system of elements $f_1, \ldots, f_m \in M$ generates M over R iff the germs $f_{1x}, \ldots, f_{mx} \in \mathcal{M}_x$ generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x} = R_x$ for every $x \in X = \mathrm{Specm}(R)$.

Let us introduce some further notations:
For $x \in X$ we denote by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal of the local ring $\mathcal{O}_{X,x}$ and by $k(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ its residue field. Further let

$$L_x(M) := \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x.$$

This is a vector space over $k(x)$. By the Lemma of Nakayama

$$d_x(M) := \dim_{k(x)} L_x(M)$$

is equal to the minimal number of generators of $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$. More precisely:

$\phi_1, \ldots, \phi_m \in \mathcal{M}_x$ generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$ iff $\phi_1(x), \ldots, \phi_m(x) \in L_x(M)$ generate $L_x(M)$ over $k(x)$.

Here we denote by $\phi_j(x)$ the image of ϕ_j under the morphism $\mathcal{M}_x \to L_x(M)$. For $f \in M$ we will denote by $f(x) \in L_x(M)$ the image of f under $M \to \mathcal{M}_x \to L_x(M)$.

The module M over R induces a certain stratification of $X = \mathrm{Specm}(R)$, which will be essential for us.

Definition. For $k \in \mathbb{N}$ let

$$X_k(M) := \{x \in X : d_x(M) \geqslant k\}.$$

It is easy to prove that $X_k(M)$ is a Zariski-closed subset of X. We have

$$X = X_0(M) \supset X_1(M) \supset \ldots \supset X_r(M) \supset X_{r+1}(M) = \emptyset.$$

where $r := \sup\{\dim_{k(x)} L_x(M) : x \in X\}$. (Since M is finitely generated, $r < \infty$.)

Let us consider some examples:

a) Suppose M is a projective module of rank r over R. Then the associated sheaf $\mathcal{M}$ is locally free of rank r (and defines by definition a vector bundle of rank r over X) . We have

$$X = X_0(M) = \ldots = X_r(M) \supset X_{r+1}(M) = \emptyset.$$

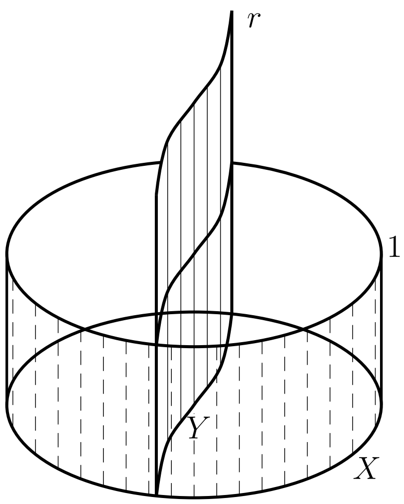
b) Let R be a regular noetherian ring and $I \subset R$ a locally complete intersection ideal of height r. If $r = 1$, the ideal I is a projective R-module (example a), so suppose $r \geqslant 2$. Let

$$X = \mathrm{Specm}(R), \quad Y = V_X(I) = \mathrm{Specm}(R/I)$$

and let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf associated to I. For $x \in X \setminus Y$, we have $\mathcal{I}_x = \mathcal{O}_{X,x}$, hence $d_x(I) = r$. For $y \in Y$, the minimal number of generators of $\mathcal{I}_Y$ equals r, hence $d_y(I) = r$. This implies

$$X = X_0(I) = X_1(I) \supset X_2(I) = \ldots = X_r(I) \supset X_{r+1}(I) = \emptyset.$$

We visualize the situation by the following picture.



We remark that the topological space $X = \mathrm{Specm}(R)$ has a certain combinatorical dimension (finite or infinite). This dimension is less or equal to the dimension of $\mathrm{Spec}(R)$, which is the Krull dimension of R. In particular, if $R = k[T_1, \ldots, T_n]$ is a polynomial ring in n indeterminates over a field, $\dim \mathrm{Specm}(R) = \dim \mathrm{Spec}(R) = n$. For a local ring R we have always $\dim \mathrm{Specm}(R) = 0$.

2.1. Theorem (Forster [9]**, Swan** [36]**).** *Let R be a noetherian ring and M a finitely generated R-module. Set*

$$b(M) := \sup\{k + \dim X_k(M) : \ k \geqslant 1, \ X_k(M) \neq \emptyset\}.$$

Then M can be generated by $b(M)$ elements.
(We set $b(M) = 0$, if $X_1(M) = \emptyset$.)

Proof by induction on $b(M)$. We may suppose $b(M) < \infty$, since otherwise there is nothing to prove.
If $b(M) = 0$, we have $\mathcal{M}_x = 0$ for all $x \in X = \mathrm{Specm}(R)$. This implies $M = 0$, hence M is generated by 0 elements.
Induction step. Let us abbreviate $X_k(M)$ by X_k. We denote by X_k^j the (finitely many) irreducible components of X_k. Let J be the set of all pairs (k, j) such that

$$k \geqslant 1 \quad \text{and} \quad k + \dim X_k^j = b(M).$$

Then $x_k^j \not\subset X_{k+1}$, since otherwise we would have $(k + 1) + \dim X_{k+1} > b(M)$, contradicting the definition of $b(M)$. Choose a point

$$x_{kj} \in X_k^j \smallsetminus X_{k+1}.$$

We have $\dim L_{x_{kj}}(M) = k > 0$ and it is easy to construct an element $f \in M$ such that

$$f(x_{kj}) \neq 0 \quad \text{for all} \quad (k, j) \in J.$$

We consider the quotient module $N := M/Rf$. By the choice of f it follows that

$$\dim L_{x_{kj}}(N) = k - 1 \quad \text{for all} \quad (k, j) \in J,$$

i.e. $x_{kj} \notin X_k(N)$. This implies $k + \dim X_k(N) < b(M)$. By induction hypothesis, N can be generated by $b(M) - 1$ elements, hence M can be generated by $b(M)$ elements. $\square$

2.2. Corollary. *Let M be a finitely generated projective module of rank r over a noetherian ring R and $n := \dim \mathrm{Specm}(R)$. Then M can be generated by $n + r$ elements.*

2.3. Corollary. *Let R be a regular noetherian ring and I be a locally complete intersection ideal of height r. Set*

$$n := \dim \mathrm{Specm}(R), \quad k := \dim \mathrm{Specm}(R/I).$$

Then I can be generated by $b(I) = \max(n + 1, k + r)$ elements.

Remark. Let $N = \dim \operatorname{Spec}(R)$ be the Krull dimension of R. Then $n \leqslant N$ and $k \leqslant N - r$, so I can always be generated by $N + 1$ elements.

We now consider the problem whether the given estimate is best possible. The answer is yes, if we make no further restrictions on M and R. In order to construct counter-examples, we need some tools from topology.

Topological vector bundles on CW-complexes

Let X be an n-dimensional CW-complex and E be a real vector bundle of rank r over X. We consider the ring $R := \mathcal{C}(X)$ of all (real-valued) continuous functions on X and the vector space $\Gamma_{\text{con}}(X, E)$ of all continuous sections of E. In a natural way, M is an R-module. Suppose M is generated by m elements over R. Then we have a module epimorphism $R^m \to M \to 0$. This corresponds to a vector bundle epimorphism

$$\theta^m \xrightarrow{\beta} E \longrightarrow 0,$$

where θ^m denotes the trivial vector bundle of rank m over X. The kernel of β is a vector bundle F of rank $m - r$ over X. The sequence

$$0 \longrightarrow F \longrightarrow \theta^m \longrightarrow E \longrightarrow 0$$

splits (use a partition of unity), so we get $\theta^m \cong E \oplus F$. We have proved:
If the module $\Gamma_{\text{con}}(X, E)$ can be generated by m elements over the ring $\mathcal{C}(X)$, then there exists a vector bundle F of rank $m - r$, such that $E \oplus F \cong \theta^m$.
It is easy to see that also the converse implication holds.

Stiefel-Whitney classes

To every real vector bundle E of rank r there are associated Stiefel-Whitney classes

$$\gamma_i(E) \in H^i(X, \mathbb{Z}_2), \quad (\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}).$$

We have $\gamma_0(E) = 1$ and $\gamma_i(E) = 0$ for $i > r$. It is convenient to consider the total Stiefel-Whitney class

$$\gamma(E) = 1 + \gamma_1(E) + \ldots + \gamma_r(E) \in H^*(X, \mathbb{Z}_2) = \bigoplus_{i \geqslant 0} H^i(X, \mathbb{Z}_2)$$

in the (commutative) cohomology ring $H^*(X, \mathbb{Z}_2)$. We will need the following properties of the Stiefel-Whitney classes (for more information see e.g. Husemoller [20]):

a) $\gamma(\theta^m) = 1$ for the trivial vector bundle θ^m.

b) If E, F are two vector bundles, then

$$\gamma(E \oplus F) = \gamma(E)\gamma(F).$$

In particular, if $E \oplus F \cong \theta^m$, then

$$\gamma(F) = \gamma(F)^{-1}.$$

We remark that every element of the form $1 + \xi_1 + \ldots + \xi_r,\ \xi_i \in H^i(X, \mathbb{Z}_2)$, is invertible in $H^*(X, \mathbb{Z}_2)$.

Example. Consider the real projective n-space $X := \mathbb{P}^n(\mathbb{R})$ as a topological space and let E be the line bundle on X corresponding to a hyperplane section. The cohomology ring of X is

$$H^*(X, \mathbb{Z}_2) \cong \mathbb{Z}_2[t]/(t^{n+1}),$$

i.e. $H^i(X, \mathbb{Z}_2) \cong \mathbb{Z}_2$ for $i = 0, \ldots, r$, and $\xi^i := t^i \operatorname{mod}(t^{n+1})$ is the non-zero element of $H^i(X, \mathbb{Z}_2)$. It is well-known that

$$\gamma(E) = 1 + \xi.$$

Suppose now that F is a vector bundle of rank $m-1$ such that $E \oplus F \cong \theta^m$. Then

$$\gamma(F) = (1 + \xi)^{-1} = 1 + \xi + \ldots + \xi^n.$$

From this follows $n \geqslant \operatorname{rank} F = m - 1$. Therefore we have proved:
The module $\Gamma_{\text{con}}(X, E)$ of sections of E cannot be generated by less than $n + 1$ elements over $\mathcal{C}(X)$.

However this example gives not yet an answer to our original problem, since the ring $\mathcal{C}(X)$ is not noetherian. But one can modify this example to construct an n-dimensional noetherian ring A and a projective A-module M of rank 1 such that the minimal number of generators is $n + 1$. For this purpose we represent projective n-space as the quotient of the n-sphere

$$S^n := \{x \in \mathbb{R}^{n+1} : x_0^2 + \ldots + x_n^2 = 1\}$$

by identifying antipodal points:

$$\mathbb{P}^n(\mathbb{R}) = S^n/\sim.$$

Let $\mathbb{R}[x_0, \ldots, x_n]^{\text{ev}}$ be the ring of all even polynomials in $x_0, \ldots, x_n$, i.e. polynomials f satisfying $f(x) = f(-x)$. We define

$$A := \mathbb{R}[x_0, \ldots, x_n]^{\text{ev}}/(x_0^2 + \ldots + x_n^2 - 1).$$

It is clear that the elements of A can be considered as continuous functions on $\mathbb{P}^n(\mathbb{R})$. The ring A is noetherian and has dimension n. The hyperplane section $x_0 = 0$ corresponds to the ideal $M \subset A$ generated by the classes of

$$x_0^2, x_0x_1, \ldots, x_0x_n.$$

M is a projective A-module of rank 1. If M is generated by m elements, we have an epimorphism $A^m \to M \to 0$. This leads to an epimorphism of vector bundles on $\mathbb{P}^n(\mathbb{R})$

$$\theta^m \longrightarrow E \longrightarrow 0,$$

where E is the line bundle corresponding to a hyperplane section. As we have proved above $m \geqslant n+1$. Thus we have got the desired example.

This example illustrates also a general theorem of Lønsted [24], which says the following: Let X be a finite n-dimensional CW-complex. Then there exists an n-dimensional noetherian ring A and a natural bijective correspondence between the isomorphism classes of real vector bundles over X and finitely generated projective modules over A.

III. Estimation of the number of global generators of a coherent sheaf on a Stein space

There exists a simple analogue of Theorem 2.1 on Stein spaces, cf. [10]. But in the analytic case the estimates can be made much better.
Let $\mathcal{M}$ be a coherent sheaf on a complex space $(X, \mathcal{O}_X)$. For $k \in \mathbb{N}$ we set

$$X_k(\mathcal{M}) := \{x \in X : \dim_{\mathbb{C}} L_x(\mathcal{M}) \geqslant k\},$$

where $L_x(\mathcal{M}) := \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x$. The $X_k(\mathcal{M})$ are analytic subsets of X.

3.1. Theorem. *Let X be a Stein space and $\mathcal{M}$ a coherent analytic sheaf on X. Set*

$$\tilde{b}(M) := \sup\left\{k + \lfloor \tfrac{1}{2} \dim X_k(\mathcal{M})\rfloor : \ k \geqslant 1,\ X_k(\mathcal{M}) \neq \emptyset\right\}.$$

Then the module $\Gamma(X, \mathcal{M})$ of global sections of $\mathcal{M}$ can be generated by $\tilde{b}(M)$ elements over the ring $\Gamma(X, \mathcal{O}_X)$.
(For $a \in \mathbb{R}$ the symbol $\lfloor a \rfloor$ denotes the greatest integer $p \leqslant a$.)

Before we come to the proof, we give some corollaries.

3.2. Corollary. *Let X be an n-dimensional Stein space and E a holomorphic vector bundle of rank r on X. Then the module $\Gamma(X, E)$ of holomorphic sections can be generated by $r + \lfloor n/2 \rfloor$ elements over $\Gamma(X, \mathcal{O}_X)$.*

In particular: Over a 1-dimensional Stein space every holomorphic vector bundle is trivial.

3.3. Corollary. *Let X be pure n-dimensional Stein space, $n \geqslant 3$, and $Y \subset X$ a curve (not necessarily reduced), which is a locally complete intersection. Then Y is a global ideal-theoretic complete intersection.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of Y. We have

$$X_1(\mathcal{I}) = X, \quad X_2(\mathcal{I}) = \ldots = X_{n-1}(\mathcal{I}) = Y, \ X_n(\mathcal{I}) = \emptyset.$$

Therefore

$$\tilde{b}(\mathcal{I}) = \max\{1 + \lfloor n/2 \rfloor, n - 1 + \lfloor 1/2 \rfloor = n - 1\}$$

since $n \geqslant 3$. By Theorem 3.1, $\Gamma(X, \mathcal{I})$ can be generated by $n-1$ elements, so Y is a complete intersection. □

Remark. The Corollary 3.3 is not valid in 2-dimensional Stein manifolds. For example in $X = \mathbb{C}^* \times \mathbb{C}^*$ there exist divisors which are not principal. Of course, in $\mathbb{C}^2$ every curve is a complete intersection.

Proof of Theorem 3.1. We have to consider only the case $\tilde{b}(\mathcal{M}) < \infty$. From this follows $\dim \operatorname{Supp}(\mathcal{M}) < \infty$. So we may suppose $\dim X < \infty$. Since our hypothesis implies that the minimal number of generators of $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$ is bounded for $x \in X$, it is relatively easy to see that there exist finitely many elements $f_1, \ldots, f_N \in \Gamma(X, \mathcal{M})$ generating this module over $\Gamma(X, \mathcal{O}_X)$.
Let $m \in \mathbb{N}$. In order to find a system of generators $g_1, \ldots, g_m \in \Gamma(X, \mathcal{M})$ consisting of m elements we make the following ansatz: Take a holomorphic $(m \times N)$-matrix

$$A = (a_{ij}) \in \mathrm{M}(m \times N, \Gamma(X, \mathcal{O}_X))$$

and define

$$g_i = \sum_{j=1}^{N} a_{ij} f_j \in \Gamma(X, \mathcal{M}) \quad \text{for} \quad i = 1, \ldots, m,$$

or in matrix notation

$$g = Af, \quad \text{where } f = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ \vdots \\ g_m \end{pmatrix}.$$

We now study the problem what conditions the matrix A has to satisfy so that $g = (g_1, \ldots, g_m)$ becomes again a system of generators.

For this purpose we define for $x \in X$ the set of matrices

$$(*) \quad E(x) := \{S \in \mathrm{M}(m \times N, \mathbb{C}) : \mathrm{rank}(Sf(x)) = d_x(\mathcal{M})\}.$$

Here $f_j(x)$ is the image of f_j in

$$L_x(\mathcal{M}) = \mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x \quad \text{and } d_x(\mathcal{M}) = \dim L_x(\mathcal{M}).$$

Note that, since $f_1, \ldots, f_N$ generate $\Gamma(X, \mathcal{M})$, the elements $f_1(x), \ldots, f_N(x)$ generate $L_x(\mathcal{M})$, hence $\mathrm{rank}(f(x)) = d_x(\mathcal{M})$ for every $x \in X$.

Claim. $g = Af$ is a system of generators of $\Gamma(X, \mathcal{M})$ iff

$$A(x) \in E(x) \quad \text{for every} \quad x \in X.$$

The necessity is clear. Suppose conversely, that $A(x) \in E(x)$ for all $x \in X$. Since $g(x) = A(x)f(x)$, we get $\mathrm{rank}(g(x)) = d_x(\mathcal{M})$, hence by the Lemma of Nakayama the germs $g_{1x}, \ldots, g_{mx}$ generate $\mathcal{M}_x$ over $\mathcal{O}_{X,x}$. Since X is Stein, this implies that $g_1, \ldots, g_m$ generate $\Gamma(X, \mathcal{M})$ over $\Gamma(X, \mathcal{O}_X)$.

We can reformulate the condition as follows. Define the following subset of the trivial bundle $X \times \mathrm{M}(m \times N, \mathbb{C})$:

$$E(\mathcal{M}, f, m) := \{(x, S) \in X \times \mathrm{M}(m \times N, \mathbb{C}) : S \in E(x)\},$$

where $E(x)$ is defined by $(*)$. It is easy to see that $E(\mathcal{M}, f, m)$ is an open subset of $X \times \mathrm{M}(m \times N, \mathbb{C})$. We have a natural projection

$$p : E(\mathcal{M}, f, m) \longrightarrow X$$

and $p^{-1}(x) = \{x\} \times E(x) \cong E(x)$. We call $E(\mathcal{M}, f, m)$ the endromis bundle of $\mathcal{M}$ with respect to the system of generators $f = (f_1, \ldots, f_N)$ and the natural number m. Note however, that in general this is not a locally trivial bundle. We have proved:

3.4. Proposition. *The module $\Gamma(X, \mathcal{M})$ can be generated by m elements over $\Gamma(X, \mathcal{O}_X)$ iff the endromis bundle $E(\mathcal{M}, f, m) \to X$ admits a holomorphic section.*

The essential tool is now an Oka principle for endromis bundles, which allows to reduce the problem to a topological problem.

3.5. Theorem (Forster/Ramspott [14]**).** *The endromis bundle $E(\mathcal{M}, f, m) \to X$ admits a holomorphic section iff it admits a continuous section.*

We cannot give a proof here, but refer to [13], [14]. It is a generalization of the Oka principle proved by Grauert [17].
It is now necessary to study some topological properties of the endromis bundle.

3.6. Proposition. *For $x \in X_x(\mathcal{M}) \smallsetminus X_{k+1}(\mathcal{M})$, the topological space $E(x)$ is homeomorphic to $W_{km} \times \mathbb{R}^t$, where W_{km} is the Stiefel manifold of orthonormal k-frames in $\mathbb{C}^m$.*

Proof. By definition we have for $x \in X_k(\mathcal{M}) \smallsetminus X_{k+1}(\mathcal{M})$

$$E(x) = \{S \in \mathrm{M}(m \times N, \mathbb{C}) : \mathrm{rank}(SF) = k\},$$

where F is a certain fixed $(N \times k)$-matrix of rank k. After a change of coordinates we may assume $F = \begin{pmatrix} \mathbb{1}_k \\ 0 \end{pmatrix}$, where $\mathbb{1}_k$ is the unit $(k \times k)$-matrix and 0 denotes the zero $((N-k) \times k)$-matrix. If we decompose $S = (S_1, S_2)$ with $S_1 \in \mathrm{M}(m \times k, \mathbb{C})$, $S_2 \in \mathrm{M}(m \times (N-k), \mathbb{C})$, then $SF = S_1$. Therefore $E(x)$ is homeomorphic to $W'_{km} \times \mathrm{M}(m \times (N-k), \mathbb{C})$, where W'_{km} is the space of all $(m \times k)$-matrices of rank k. But W'_{km} is up to a factor $\mathbb{R}^s$ homeomorphic to the Stiefel manifold W_{km}. □

More precisely one can prove:

3.7. Proposition. *$E(\mathcal{M}, f, m)|_{X_k(\mathcal{M}) \smallsetminus X_{k+1}(\mathcal{M})}$ is a locally trivial fibre bundle with fibre homeomorphic to $W_{km} \times \mathbb{R}^t$.*

To be able to apply topological obstruction theory to the endromis bundle, we have to know some homotopy groups of the Stiefel manifolds.

3.8. Proposition. $\pi_q(W_{km}) = 0$ *for all* $q \leqslant 2(m-k)$.

Proof by induction on k.

$k = 1$. The Stiefel manifold W_{1m} is nothing else than the $(2m-1)$-sphere S^{2m-1}, hence $\pi_q(W_{1m}) = 0$ for $q \leqslant 2(m-1)$.

$k-1 \to k$. By associating to a k-frame its first vector, we get a fibering

$$W_{k-1,m-1} \rightarrowtail W_{km} \twoheadrightarrow S^{2m-1},$$

hence an exact homotopy sequence

$$\ldots \longrightarrow \pi_{q+1}(S^{2m-1}) \longrightarrow \pi_q(W_{k-1,m-1}) \longrightarrow \pi_q(W_{km}) \longrightarrow \pi_q(S^{2m-1}).$$

For $q < 2m-1$ we have therefore isomorphisms $\pi_q(W_{k-1,m-1}) \cong \pi_q(W_{km})$. By induction hypothesis the assertion follows. □

We will apply the following theorem of obstruction theory for fibre bundles (cf. Steenrod [32]):

3.9. Theorem. *Let X be a CW-complex, Y a subcomplex and $E \to X$ a locally trivial fibre bundle with typical fibre F and connected structure group. Let $s : Y \to E$ be a section of E over Y. If*

$$H^{q-1}(X, Y; \pi_q(F)) = 0 \quad \textit{for all} \quad q \geqslant 1,$$

then there exists a global section $\overline{s} : X \to E$ with $\overline{s}|_Y = s$.

This theorem can in particular be applied to complex spaces with countable topology since these spaces can be triangulated (Giesecke [15], Łojasiewicz [23]). Note that every connected component of a Stein space has countable topolgy (Grauert [16]).

Theorem 3.1 will now be a consequence of the following proposition.

3.10. Proposition. *If $m \geqslant \tilde{b}(\mathcal{M})$, then the endromis bundle $E(\mathcal{M}, f, m) \to X$ admits a continuous section.*

Proof. Let $r = \sup_{x \in X} \dim L_x(\mathcal{M})$ and write x_k for $x_k(\mathcal{M})$. We have

$$X = X_0 \supset X_1 \supset \ldots \supset X_r \supset X_{r+1} = \emptyset.$$

We construct a section $s_k : X_k \to E(\mathcal{M}, f, m)$ by descending induction on k.

$k = r$. $E(\mathcal{M}, f, m)|_{X_r}$ is a locally trivial fibre bundle with fibre homotopically equivalent to W_{rm}. The obstructions to finding a section lie in

$$H^{q+1}(x_r, \pi_q(W_{rm})), \quad q \geqslant 1.$$

By Proposition 3.8 we have only to consider the case $q \geqslant 2(m - r) + 1$. Since $m \geqslant \tilde{b}(\mathcal{M})$, we have in particular

$$r + \lfloor \tfrac{1}{2} \dim X_r \rfloor \leqslant m,$$

hence $\dim X_r \leqslant 2(m-1)+1$. But for an arbitrary Stein space Z and an arbitrary abelian group G we have

$$H^{q+1}(Z, G) = 0 \quad \text{for all} \quad q \geqslant \dim Z$$

(Theorem of Andreotti-Frankel [1], Hamm [19]). Thus $H^{q+1}(X_r, \pi_q(W_{rm})) = 0$ for all $q \geqslant 1$ and the section $s_r : X_r \to E(\mathcal{M}, f, m)$ can be constructed.

$k+1 \to k$. From Proposition 3.8 and the theorem of Andreotti-Frankel-Hamm we conclude again that

$$H^{q+1}(X_k, X_{k+1}; \pi_q(W_{km})) = 0 \quad \text{for all} \quad q \geqslant \dim 1.$$

This will allow us to extend the section $s_{k+1} : X_{k+1} \to E(\mathcal{M}, f, m)$ over X_k. However, we cannot apply Theorem 3.9 directly, since $E(\mathcal{M}, f, m)$ is not a locally trivial fibre bundle. So we proceed as follows: We first extend the section s_{k+1} to a section $\tilde{s}$ over a small neighborhood T of X_{k+1} in X_k. We can choose T in such a way that X_{k+1} is a deformation retract of T. Over $X_k \smallsetminus X_{k+1}$ the endromis bundle is locally trivial and we can apply Theorem 3.9 to extend the section $\tilde{s}|_{T \smallsetminus X_{k+1}}$ over all of $X_k \smallsetminus X_{k+1}$. This is possible, since the relative cohomology of the pair $(X_k \smallsetminus X_{k+1}, T_k \smallsetminus X_{k+1})$ is the same as of the pair (X_k, X_{k+1}). □

The technique of endromis bundles also allows to prove the following theorem.

3.11. Theorem ([14]). *Let $Y \subset \mathbb{C}^n$ be a pure m-dimensional locally complete intersection with $m \leqslant \frac{2}{3}(n-1)$. Then Y is a (global, ideal-theoretic) complete intersection if and only if the conormal bundle of Y is trivial.*

Proof. Let $\mathcal{I} \subset \mathcal{O}_{\mathbb{C}^n}$ be the ideal sheaf of Y. The conormal bundle of Y is given by $\mathcal{I}/\mathcal{I}^2$, which is a locally free sheaf of rank $r = n - m$ over $\mathcal{O}_Y = \mathcal{O}_{\mathbb{C}^n}/\mathcal{I}$. If Y is a complete intersection, $\Gamma(\mathbb{C}^n, \mathcal{I})$ is generated by r elements over $\Gamma(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$. Then also $\Gamma(Y, \mathcal{I}/\mathcal{I}^2)$ is generated by r elements over $\Gamma(Y, \mathcal{O}_Y)$, hence $\mathcal{I}/\mathcal{I}^2$ is free, i.e. the conormal bundle of Y is trivial.

Conversely, suppose that the conormal bundle is trivial. Then there exist functions $f_1, \ldots, f_r \in \Gamma(\mathbb{C}^n, \mathcal{I})$, whose classes modulo $\mathcal{I}^2$ generate $\Gamma(Y, \mathcal{I}/\mathcal{I}^2)$. Therefore the germs $f_{1x}, \ldots, f_{rx}$ generate $\mathcal{I}_x$ for all x in some neighborhood of Y.

Consider now the endromis bundle $E = E(\mathcal{I}, g, r) \to \mathbb{C}^n$ for some system of generators $g = (g_1, \ldots, g_N)$ of $\Gamma(\mathbb{C}^n, \mathcal{I})$. The functions $f_1, \ldots, f_r$ give rise to a section s of E over some neighborhood of Y. We have to extend this section continuously over $\mathbb{C}^n$. As in the proof of Proposition 3.10, the obstructions to this extension lie in $H^{q+1}(\mathbb{C}^n, Y; \pi_q(W_{1r}))$. The Hypothesis $m \leqslant \frac{2}{3}(n-1)$ implies $2r - 1 > m = \dim Y$, hence by the theorem of Andreotti-Frankel-Hamm the groups

$$H^{q+1}(\mathbb{C}^n, Y; \pi_q(W_{1r})) \simeq H^q(Y, \pi_q(S^{2r-1}))$$

vanish. □

IV. Theorems of Mohan Kumar

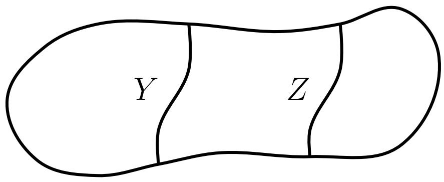
In the algebraic case one cannot apply the strong tools of algebraic topology as in the theory of Stein spaces. One has to use other methods. We expose here some results of Mohan Kumar [25], [26].

We begin with a simple proposition.

4.1. Proposition. *Let R be a noetherian ring and $I \subset R$ an ideal. If I/I^2 can be generated by m elements over R/I, then I can be generated by $m+1$ elements over R.*

Proof. Let $X = \mathrm{Spec}(R)$ be the affine scheme associated to R and $Y = V(I) \subset X$ the subspace defined by I. We denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf associated to I. Let $f_1, \ldots, f_m \in I$ be elements generating $I \bmod I^2$. By the Lemma of Nakayama the germs $f_{1x}, \ldots, f_{mx}$ generate the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ for all $x \in Y$ and by coherence this is true even for all x in a certain neighborhood of Y. Therefore

$$V(f_1, \ldots, f_m) = Y \cup Z,$$



$X = \mathrm{Spec}(R)$

where $Z \subset X$ is a closed subset disjoint from Y. In particular we have that $f_{1x}, \ldots, f_{mx}$ generate $\mathcal{I}_x$ for all $x \in X \setminus Z$. Now there exists a function $f_{m+1} \in \Gamma(X, \mathcal{O}_X) = R$ such that $f_{m+1}|_Z = 1$ and $f_{m+1}|_Y = 0$ (i.e. $f_{m+1} \in I$). Then $f_1, \ldots, f_{m+1}$ generate I over R (since this is true locally).

□

4.2. Theorem (Mohan Kumar [25]**).** *Let Y be a smooth pure m-dimensional algebraic subvariety in affine n-space $\mathbb{A}^n$ (over an algebraically closed field). Suppose $2m+1 < n$. Then Y is a complete intersection (in the ideal theoretic sense) if and only if the normal bundle of Y is trivial.*

Remark. This theorem is only a special case of the next theorem. We will prove it here, since the method of proof is interesting for itself.

He have to recall some notions of algebraic K-theory.

Definition. Two vector bundles E, F over an algebraic variety X are called *stably isomorphic*, if there exist trivial bundles θ^k, θ^l over X such that $E \oplus \theta^k \cong F \oplus \theta^l$. A vector bundle E is called stably *trivial*, if it is stably isomorphic to a trivial bundle.

One has the following

Cancellation Theorem. *Let E and F be stably isomorphic vector bundles of the same (constant) rank r over an n-dimensional affine algebraic variety X. If $r \geqslant n+1$, then E and F are isomorphic.*

More generally, this Cancellation Theorem holds for projective modules over n-dimensional noetherian rings, see e.g. Bass [3].

Remark. The same Cancellation Theorem holds also in the topological category for real vector bundles:
If E, F are two stably isomorphic real vector bundles over an n-dimensional CW-complex X and if $r \geqslant n+1$, then E and F are topologically isomorphic.

For complex vector bundles, Cancellation is already possible for $r \geqslant n/2$. The Oka principle for holomorphic vector bundles on Stein spaces then implies the following: Let E, F be two holomorphic vector bundles of rank r over an n-dimensional Stein space X. Suppose $r \geqslant n/2$. If E and F are stably isomorphic, they are analytically isomorphic.

Proof of Theorem 4.2. Since $n > 2\dim Y + 1$, we can choose coordinates in $\mathbb{A}^n$ such that, denoting by $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ the projection to the first $n-1$ coordinates, p maps, Y isomorphically onto a smooth algebraic subvariety $Y' \subset \mathbb{A}^{n-1}$.
Over Y we have the exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_{\mathbb{A}^n}|_Y \longrightarrow N_{Y/\mathbb{A}^n} \longrightarrow 0,$$

where T stands for the tangent bundle and N for the normal bundle. Since Y is affine, the sequence splits. Thus we have

$$T_Y \oplus N_{Y/\mathbb{A}^n} = \theta^n,$$

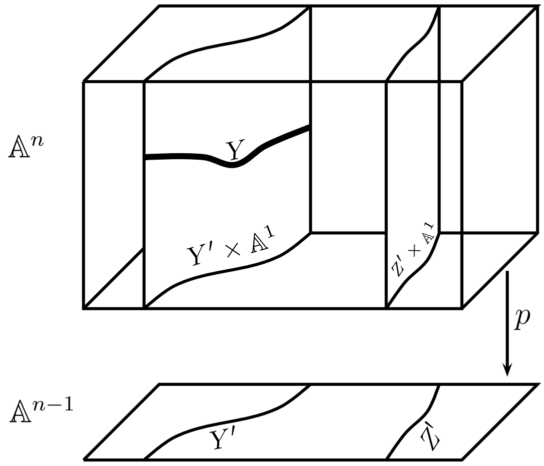
where $\theta^n = T_{\mathbb{A}^n}|_Y$ is the trivial n-bundle over Y.

Now we suppose that the normal bundle of Y is trivial. This implies that the tangent bundle T_Y is stably trivial. Because $Y \cong Y'$, also the tangent bundle $T_{Y'}$ is stably trivial. From the isomorphism $T_{Y'} \oplus N_{Y'/\mathbb{A}^{n-1}} = \theta^{n-1}$ we conclude then that $N_{Y'/\mathbb{A}^{n-1}}$ is stably trivial. But $\operatorname{rank} N_{Y'/\mathbb{A}^{n-1}} = (n-1) - m > m = \dim Y'$, so by the Cancellation Theorem $N_{Y'/\mathbb{A}^{n-1}}$ is in fact trivial. This means that $I_{Y'}/I_{Y'}^2$ is a free module of rank $r-1$ over $\Gamma(Y, \mathcal{O}_{Y'}) = K[T_1, \ldots, T_{n-1}]/I_{Y'}$, where $r = n-m$. Choose polynomials $f_1, \ldots, f_{r-1} \in K[T_1, \ldots, T_{n-1}]$ which generate $I_{Y'} \bmod I_{Y'}^2$. We have

$$V_{\mathbb{A}^{n-1}}(f_1, \dots, f_{r-1}) = Y' \cup Z',$$

where $Z' \subset \mathbb{A}^{n-1}$ is an algebraic subset disjoint from Y' (cf. the proof of Proposition 4.1). We can consider the f_j also as elements of $K[T_1, \dots, T_n]$ and have

$$V_{\mathbb{A}^n}(f_1, \dots, f_{r-1}) = (Y' \times \mathbb{A}^1) \cup (Z' \times \mathbb{A}^1).$$



Since $p|_Y \to Y$ is an isomorphism, Y is a graph over Y', hence there exists an element $\varphi \in \Gamma(Y', \mathcal{O}_{Y'})$ such that the ideal of Y in

$$\Gamma(Y' \times \mathbb{A}^1, \mathcal{O}_{Y' \times \mathbb{A}^1}) = \Gamma(Y', \mathcal{O}_{Y'})[T_n]$$

is generated by $\Phi := T_n - \varphi$. Now choose an element $f_r \in K[T_1, \dots, T_n]$ such that

$$f_r|_{Y' \times \mathbb{A}^1} = \Phi, \quad \text{and} \quad f_r|_{Z' \times \mathbb{A}^1} = 1.$$

phahhas Then $f_1, \dots, f_r$ generate the ideal of Y in $K[T_1, \dots, T_n]$, hence Y is a complete intersection. $\square$

Problem. Let $Y \subset \mathbb{A}^n$ be a smooth subvariety (or a locally complete intersection) with trivial normal bundle. Can one conclude that Y is a complete intersection without the dimension restriction $2 \dim Y + 1 < n$ of Mohan Kumar's theorem?

In the case $\operatorname{codim} Y \leqslant 2$ this is always true (cf. Sec. 5). The simplest case that remains open are surfaces in $\mathbb{A}^5$.

4.3. Theorem (Mohan Kumar [26]**).** *Let $I \subset K[T_1, \dots, T_n]$ be an ideal such that I/I^2 is generated by s elements (K arbitrary field). If $s > m + 1$, where $m = \dim V(I)$, then also I can be generated by s elements.*

Remark. If we take I to be the ideal of a locally complete intersection with trivial normal bundle, we get a generalization of Theorem 4.2 to locally complete intersections.

Proof. We set $Y = V(I) \subset \mathbb{A}^n$.
We first reduce the general case to the case $\operatorname{codim} Y \geqslant 2$. If Y contains components of dimension $n-1$, then one can write $I = h \cdot J$, where $\operatorname{codim} V(J) \geqslant 2$ and $h \in K[T_1, \ldots, T_n]$ is a generator of the intersection of all primary components of dimension $n-1$ of I. The ideals I and J have the same number of generators.

So we may suppose $m = \dim Y \leqslant n-2$. After a change of coordinates we may suppose that I contains a monic polynomial g with respect to T_n. If $p : \mathbb{A}^n \to \mathbb{A}^{n-1}$ denotes the projection to the first $n-1$ coordinates, then $p|_{V(g)} \to \mathbb{A}^{n-1}$ is proper, in particular $p|_Y \to \mathbb{A}^{n-1}$ is proper. Therefore

$$Y' := p(Y) \subset \mathbb{A}^{n-1}$$

is an algebraic subset of dimension $m < n-1$.

By hypothesis, I/I^2 can be generated by s elements. Let $f_1, \ldots, f_s \in I \subset K[T_1, \ldots, T_n]$ be representatives of a system of generators of I/I^2. By adding suitable elements of I^2, we may suppose that

a) f_1 is monic with respect to T_n.
(If this is not the case, add a sufficiently high power of g.)

b) $V(f_1, \ldots, f_s) \cap (Y' \times \mathbb{A}^1) = Y$.
(This is possible since $\dim(Y' \times \mathbb{A}^1) = m + 1 < s$, by an argument similar to the proof of Theorem 1.1.)

Write $V(f_1, \ldots, f_s) = Y \cup Z$, $Z \cap Y = \emptyset$. By condition a), $p|_Z \to \mathbb{A}^{n-1}$ is proper, hence $Z' := p(Z)$ is an algebraic subset of $\mathbb{A}^{n-1}$ and by condition b) we have $Y' \cap Z' = \emptyset$. There exist affine open subsets $U',\ V' \subset \mathbb{A}^{n-1}$ such that

$$Y' \subset U' \subset \mathbb{A}^{n-1} \smallsetminus Z',$$
$$Z' \subset V' \subset \mathbb{A}^{n-1} \smallsetminus Y'$$

and $U' \cap V' = \mathbb{A}^{n-1}$. Let $U := p^{-1}(U'),\ V := p^{-1}(V')$.

Denote by $\mathcal{I} \subset \mathcal{O}_{\mathbb{A}^n}$ the ideal sheaf associated to I. Since $U \cap Z = \emptyset$, we get an exact sequence

$$(1) \qquad \mathcal{O}^s_{\mathbb{A}^n} \xrightarrow{(f_1,\ldots,f_s)} \mathcal{I} \longrightarrow 0 \qquad \text{over} \quad U.$$

Since $V \cap Y = \emptyset$, we have $\mathcal{I}|_V = \mathcal{O}_{\mathbb{A}^n}|_V$, hence an exact sequence

$$(2) \qquad \mathcal{O}^s_{\mathbb{A}^n} \xrightarrow{(1,0,\ldots,0)} \mathcal{I} \longrightarrow 0 \qquad \text{over} \quad V.$$

We want to patch together these two sequences over $U \cap V$, which is affine algebraic. To do this, we remark that $(f_1 \ldots, f_s)|_{U\cap V}$ generates the unit ideal of the ring

$$\Gamma(U \cap V, \mathcal{O}_{\mathbb{A}^n}) = \Gamma(U' \cap V', \mathcal{O}_{\mathbb{A}^{n-1}})[T_n] =: A[T_n]$$

and f_1 is monic with respect to T_n. Therefore by a theorem of Quillen-Suslin [28], [34] $(f_1 \ldots, f_s)|_{U\cap V}$ can be completed to an invertible $(s \times s)$-matrix

$$F \in \mathrm{Gl}(s, A[T_n]),$$

whose first row is $(f_1, \ldots, f_s)|_{U\cap V}$. This matrix defines an isomorphism $F : \mathcal{O}^s_{\mathbb{A}^n} \to \mathcal{O}^s_{\mathbb{A}^n}$ over $U \cap V$ and we get a commutative diagram

$$\begin{tikzcd} \mathcal{O}^s_{\mathbb{A}^n} \arrow[r, "(f_1,\ldots,f_s)"] \arrow[d, "F"'] & \mathcal{I} \arrow[d, equal] \\ \mathcal{O}^s_{\mathbb{A}^n} \arrow[r, "(1,0,\ldots,0)"] & \mathcal{I} \end{tikzcd} \qquad \text{over} \quad U \cap V.$$

Let $\mathcal{M}$ be the locally free module sheaf over $\mathbb{A}^n$ obtained by glueing $\mathcal{O}^s_{\mathbb{A}^n}|_U$ and $\mathcal{O}^s_{\mathbb{A}^n}|_V$ over $U \cap V$ by means of the isomorphism F. The sequences (1) and (2) now patch together to a single exact sequence

$$\mathcal{M} \longrightarrow \mathcal{I} \longrightarrow 0 \quad \text{over} \quad \mathbb{A}^n.$$

Again by Quillen-Suslin's solution of the Serre problem, the sheaf $\mathcal{M}$ is free, i.e. globally isomorphic to $\mathcal{O}^s_{\mathbb{A}^n}$. This means that $I = \Gamma(\mathbb{A}^n, \mathcal{I})$ can be generated by s elements. $\square$

4.4. Corollary. *Let $Y \subset \mathbb{A}^n$ be a locally complete intersection (not necessarily of pure dimension). Then the ideal I of Y can be generated by n elements.*

(For smooth Y this was a conjecture of Forster [9].)

Proof. As in the proof of Theorem 4.3, we may suppose that $m := \dim Y \leqslant n-2$. Now we can apply Theorem 4.3, if we have proved that I/I^2 can be generated by n elements. But this follows from Theorem 2.1. $\square$

Mohan Kumar [26] has also proved by similar techniques a conjecture of Eisenbud-Evans [6], which is a generalization of Corollary 4.4. (It was also proved by Sathaye [29] under some restrictions.) This can be formulated as follows:

Let A be a noetherian ring of finite Krull dimension, $R := A[T]$ and M a finitely generated R-module. Let $X := \mathrm{Spec}(R)$ and define in analogy to Sec. II

$$X_k(M) = \{x \in X : \dim_{k(x)}(\mathcal{M}_x/\mathfrak{m}_x\mathcal{M}_x) \geqslant k\},$$

where $\mathcal{M}$ is the module sheaf on X associated to M. Let

$$b_k^*(M) := \begin{cases} 0 & \text{if } X_k(M) = \emptyset, \\ k + \dim_k(M) & \text{if } 0 \leqslant \dim X_k(M) < \dim X, \\ k + \dim X_k(M) - 1, & \text{if } \dim X_k(M) = \dim X. \end{cases}$$

Then M can be generated by

$$b^*(M) := \sup\{b_k^*(M) : k \geqslant 1\}$$

elements.

V. Set-theoretic complete intersections

By the theorem of Mohan Kumar, an m-dimensional locally complete intersection Y with trivial normal bundle in affine n-space is a complete intersection, if $2m+2 \leqslant n$. Boratyński [4] has proved that without any restriction on the dimension, Y is at least a set-theoretic complete intersection.

5.1. Theorem (Boratyński). *Let $Y \subset \mathbb{A}^n$ (resp. $Y \subset \mathbb{C}^n$) be an algebraic (resp. analytic) locally complete intersection with trivial normal bundle. Then Y is a set-theoretic complete intersection.*

Proof. Set $X = \mathbb{A}^n$ (resp. $X = \mathbb{C}^n$). Let $r = \mathrm{codim}\, Y$ and $f_1, \ldots, f_r \in \Gamma(X, \mathcal{I}_Y)$ functions such that the classes $f_j \bmod \mathcal{I}_Y^2$ form a global basis of the conormal bundle $\nu_{Y/X} = \mathcal{I}_Y/\mathcal{I}_Y^2$. Then the zero set of $f_1, \ldots, f_r$ can be written as

$$V(f_1, \ldots, f_r) = Y \cup Y',$$

where $Y' \subset X$ is an algebraic (analytic) set disjoint from Y. The sets Y, Y' are contained in disjoint hypersurfaces,

$$Y \subset H, \quad Y' \subset H', \quad H \cap H' = \emptyset.$$

The set $U := X \smallsetminus (H \cup H')$ is affine algebraic (resp. Stein). Since $f_1, \ldots, f_r$ have no common zeros on U, they generate the unit ideal in the ring $\Gamma(U, \mathcal{O}_X)$. By a theorem of Suslin [35] there exists a matrix $F \in \mathrm{Gl}(r, \Gamma(U, \mathcal{O}_X))$, whose first row is $(f_1, \dots, f_{r-1}, f_r^{(r-1)!})$. Consider the ideal $\mathcal{I}_Z \subset \mathcal{O}_X$ generated by $(f_1, \dots, f_{r-1}, f_r^{(r-1)!})$ over $X \smallsetminus H'$ and equal to $\mathcal{O}_X$ over $X \smallsetminus H$. Then $V(\mathcal{I}_Z) = V(\mathcal{I}_Y) = Y$. The vectors

$$\varphi := (f_1, \dots, f_{r-1}, f_r^{(r-1)!}), \quad \psi := (1, 0, \dots, 0)$$

define epimorphisms

$$\begin{aligned} \varphi : \mathcal{O}_X^r &\longrightarrow \mathcal{I}_Z \longrightarrow 0 \quad \text{over} \quad X \smallsetminus H', \\ \psi : \mathcal{O}_X^r &\longrightarrow \mathcal{I}_Z \longrightarrow 0 \quad \text{over} \quad X \smallsetminus H. \end{aligned}$$

The matrix F defines an isomorphism $F : \mathcal{O}_X^r \to \mathcal{O}_X^r$ over $X \smallsetminus (H \cup H')$ such that $\varphi = \psi \circ F$. Therefore, denoting by $\mathcal{M}$ the locally free sheaf on X obtained by glueing $\mathcal{O}_X^r|_{X \smallsetminus H'}$ and $\mathcal{O}_X^r|_{X \smallsetminus H}$ over $X \smallsetminus (H \cup H')$ by means of F, we get an epimorphism

$$\mathcal{M} \longrightarrow \mathcal{I}_Z \longrightarrow 0 \quad \text{over} \quad X.$$

Since $X = \mathbb{A}^n$ (resp. $X = \mathbb{C}^n$), the sheaf $\mathcal{M}$ is globally free of rank r, i.e. Z is a complete intersection. Hence Y is a set-theoretic complete intersection. □

Remark. If the codimension $r = 2$, then Y is an ideal-theoretic complete intersection, since in this case $\mathcal{I}_Z = \mathcal{I}_Y$.

The Ferrand construction

If the conormal bundle of a locally complete intersection is not trivial, one can try to change the structure of the subvariety by adding nilpotent elements in order to make the conormal bundle trivial. Such a device has been invented by Ferrand [8] and Szpiro [37] to prove that locally complete intersection curves in $\mathbb{A}^3$ are set-theoretic complete intersections.

Let Y be a locally complete intersection in a complex manifold X (resp. smooth algebraic variety) with conormal bundle $\nu_{Y/X} = \mathcal{I}_Y/\mathcal{I}_Y^2$. Suppose there is given a line bundle L on Y and an epimorphism $\beta : \nu_{Y/X} \to L$. Then we can define a new ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ with $\mathcal{I}_Y^2 \subset \mathcal{I}_Z \subset \mathcal{I}_Y$ by the exact sequence

$$(1) \quad 0 \longrightarrow \mathcal{I}_Z/\mathcal{I}_Y^2 \longrightarrow \mathcal{I}_Y/\mathcal{I}_Y^2 \xrightarrow{\beta} L \longrightarrow 0.$$

Then $Z = (|Y|, \mathcal{O}_X/\mathcal{I}_Z)$ is again a locally complete intersection. This can be seen as follows: For $y \in Y$ 1et $f_1, \dots, f_r \in \mathcal{I}_{Y,y}, r = \operatorname{codim}_y Y$, be a minimal system of generators. Then the classes $[f_j] := f_j \operatorname{mod} \mathcal{I}_{Y,y}^2$ form a basis of $(\mathcal{I}_Y/\mathcal{I}_Y^2)_y$ over $\mathcal{O}_{Y,y}$. We can choose the f_j in such a way that $[f_1], \dots, [f_{r-1}]$ generate the kernel of β. Then

$$\mathcal{I}_{Z,y} = (f_1, \dots, f_{r-1}) + \mathcal{I}_{Y,y}^2 = (f_1, \dots, f_{r-1}, f_r^2).$$

Conormal bundle of Z

Since Z is again a locally complete intersection, $\nu_{Z/X} = \mathcal{I}_Z/\mathcal{I}_Z^2$ is a locally free sheaf over $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$. We consider its analytic restriction to Y,

$$\nu_{Z/X}|_Y = (\mathcal{I}_Z/\mathcal{I}_Z^2) \otimes (\mathcal{O}_X/\mathcal{I}_Y) = \mathcal{I}_Z/\mathcal{I}_Y\mathcal{I}_Z,$$

which fits into an exact sequence

$$(2) \quad 0 \longrightarrow \mathcal{I}_Y^2/\mathcal{I}_Y\mathcal{I}_Z \longrightarrow \mathcal{I}_Z/\mathcal{I}_Y\mathcal{I}_Z \longrightarrow \mathcal{I}_Z/\mathcal{I}_Y^2 \longrightarrow 0.$$

Using the isomorphisms $L \cong \mathcal{I}_Y/\mathcal{I}_Z$ and $L^2 \cong \mathcal{I}_Y^2/\mathcal{I}_Y\mathcal{I}_Z$, we can combine the exact sequences (1) and (2) to obtain the exact sequence

$$0 \longrightarrow L^2 \longrightarrow \nu_{Z/X}|_Y \longrightarrow \nu_{Y/X} \longrightarrow L \longrightarrow 0.$$

From this follows in particular

$$\det(\nu_{Z/X}|_Y) \cong \det(\nu_{Y/X}) \otimes L.$$

5.2. Theorem. *Let $Y \subset \mathbb{A}^n$ be a curve, which is a locally complete intersection. Then Y is a set-theoretic complete intersection.*

This theorem is due to Szpiro for $n = 3$ (cf. [37]) and to Mohan Kumar [26] for $n > 3$.

Proof. For $n < 3$ the theorem is trivial, so suppose $n \geqslant 3$.
Let $\nu_{Y/X}$ be the conormal bundle of Y in $X := \mathbb{A}^n$ and set $L := \det(\nu_{Y/X})^*$. The bundle $E := \nu_{Y/X}^* \otimes L$ has rank $n - 1 \geqslant 2$. Since Y is affine algebraic and 1-dimensional, E admits a section without zeros. This section corresponds to an epimorphism $\beta : \nu_{Y/X} \to L$. Applying the Ferrand construction, we get a new structure Z on $|Y|$ such that

$$\det(\nu_{Z/X}|_Y) \cong \det(\nu_{Y/X}) \otimes L \cong \mathcal{O}_Y.$$

A vector bundle on a 1-dimensional affine algebraic space is already determined by its determinant. Therefore $\nu_{Z/X}|_Y$ is trivial, hence also $\nu_{Z/X}$ is trivial. The assertion follows by applying Theorem 5.1. □

Remark. Cowsik-Nori [5] have proved, that in affine n-space over a field of characteristic $p > 0$ every curve is a set-theoretic complete intersection. But the proof cannot be carried over to characteristic zero.

5.3. Theorem **([27],[2],[30]).** *Let $Y \subset \mathbb{C}^n$ be an analytic subspace which is a locally complete intersection of (pure) dimension $m \leqslant 3$. Then Y is a set-theoretic complete intersection.*

Proof. We proceed as in the proof of Theorem 5.2. That in the analytic case Y may have dimension up to 3, is due to the following facts on vector bundles over Stein spaces.

5.4. Proposition. *Let E be a holomorphic vector bundle of rank r over an m-dimensional Stein space Y. If $r > m/2$, then E admits a holomorphic section without zeros.*

Proof. Let E_0 be the bundle with fibre $\mathbb{C}^r \smallsetminus 0$ obtained by deleting the zero section from E. By the Oka principle, it suffices to construct a continuous section of E_0. The obstructions lie in

$$H^{q+1}(Y, \pi_q(S^{2r-1})).$$

Since $2r - 1 \geqslant \dim Y$, these groups vanish by the theorem of Andreotti-Frankel-Hamm. □

5.5. Proposition. *Let E be a holomorphic vector bundle of rank r over a Stein space Y of dimension $m \leqslant 3$. If the line bundle* $\det(E)$ *is trivial, then E is trivial itself.*

Proof. By multiple application of Proposition 5.4 one gets

$$E \cong L \oplus \theta^{r-1},$$

where L is a line bundle and θ^{r-1} the trivial vector bundle of rank $r - 1$ over X. But then $\det(E) \cong L$. If this is trivial, E must be trivial. □

We refer to [2], [30], [31] for more results on (ideal-theoretic and set-theoretic) complete intersections in Stein spaces.

Problem (Murthy). Is every locally complete intersection in $\mathbb{C}^n$ (resp. $\mathbb{A}^n$) a set-theoretic complete intersection?

In order to make substantial progress in the problem of set-theoretic complete intersections it seems necessary to devise new techniques (besides the Ferrand construction) to change the structure of subvarieties and influence their conormal bundle.

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