

# Optimal Numeraires for Risk Measures\*

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## Abstract

Can the usage of a risky numeraire with a greater than risk free expected return reduce the capital requirements in a solvency test? I will show that this is not the case. In fact, under a reasonable technical condition, there exists no optimal numeraire which yields smaller capital requirements than any other numeraire.

## 1 Statement and Proof of the Result

Can the usage of a risky numeraire with a greater than risk free expected return reduce the capital requirements in a solvency test? I will show that this is not the case. In fact, under a reasonable technical condition, there exists no optimal numeraire which yields smaller capital requirements than any other numeraire.

We consider a one period setup. Terminal nominal values are modelled as essentially bounded random variables  $X \in L^\infty$  on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Random variables that coincide almost surely are identified. The riskiness of a portfolio is quantified by a convex risk measure  $\rho$  on  $L^\infty$  satisfying the following “coherence” axioms (introduced by Artzner et al. [1] and further extended to the convex case by Föllmer and Schied [5, 6]):

$$\text{convexity: } \rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y) \quad \text{for } \lambda \in [0, 1], \quad (1)$$

$$\text{monotonicity: } \rho(X) \geq \rho(Y) \quad \text{if } X \leq Y, \quad (2)$$

$$\text{cash-invariance: } \rho(X + m) = \rho(X) - m \quad \text{for } m \in \mathbb{R}, \quad (3)$$

$$\text{normality: } \rho(0) = 0. \quad (4)$$

It is legitimate practice to discount terminal values by a numeraire — one euro tomorrow is less than one euro today. We denote by  $r \geq 0$  the prevailing risk free rate. The regulatory required capital (the “solvency capital

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requirement”) an insurance company must have available at the beginning of the accounting period is

$$\rho(X/e^r - x) = x + \rho(X/e^r), \quad (5)$$

where  $x \in \mathbb{R}$  and  $X \in L^\infty$  denote initial and terminal nominal value of the company’s portfolio, respectively. That is,  $\rho(X/e^r)$  equals the amount of risk free bonds the company needs in addition (can withdraw, if negative) at inception to become (remain) acceptable.

Can we replace the risk free bond by a risky numeraire and achieve a reduction of capital requirements? Indeed, let  $U > 0$  denote the terminal nominal value of a traded financial instrument. Since used as a numeraire, we can normalize it and assume that its initial value is one. The required capital becomes  $x + \rho(X/U)$ . Obviously, one would chose a numeraire with a greater than risk free expected return, i.e.  $\mathbb{E}[U] > e^r$ . However, it turns out that there is no optimal numeraire, as the following theorem indicates:

**Theorem 1.1.** *Assume that  $\rho$  is sensitive, that is,*

$$\rho(-1_A) > 0 \quad \text{for all } A \in \mathcal{F} \text{ with } \mathbb{P}[A] > 0. \quad (6)$$

Let  $U, V > 0$  be two random variables and denote

$$\mathcal{M} := \{Z \mid Z/U \in L^\infty \text{ and } Z/V \in L^\infty\}.$$

Then  $\rho(Z/U) \leq \rho(Z/V)$  for all  $Z \in \mathcal{M}$  if and only if  $U = V$ .

*Proof.* Sufficiency of the statement is clear.

To prove necessity, we first recall the well known representation result for convex risk measures on  $L^\infty$  (see e.g. [6] or [3]). Let  $(L^\infty)^*$  denote the dual space of  $L^\infty$ , that is, the space of bounded finitely additive measures  $\nu$  which are absolutely continuous with respect to  $\mathbb{P}$ . We define the convex set

$$\mathcal{C} := \{\nu \in (L^\infty)^* \mid \langle \nu, 1 \rangle = -1 \text{ and } \langle \nu, Y \rangle \leq 0 \text{ for all } Y \geq 0\}. \quad (7)$$

Then, for all  $Y \in L^\infty$ ,

$$\rho(Y) = \max_{\nu \in \mathcal{C}} (\langle \nu, Y \rangle - \rho^*(\nu)) \quad (8)$$

where  $\rho^*$  denotes the convex conjugate of  $\rho$ , which, in view of (4), is positive:

$$\rho^*(\nu) = \sup_{Z \in L^\infty} \langle \nu, Z \rangle - \rho(Z) \geq \langle \nu, 0 \rangle - \rho(0) = 0. \quad (9)$$

Now let  $n \in \mathbb{N}$  and denote  $A_n := \{1/n \leq U < V \leq n\}$ . We argue by contradiction and suppose  $\mathbb{P}[A_n] > 0$ . Clearly,  $Z := -V1_{A_n} \in \mathcal{M}$ . The above results (8), (9) and (6) therefore imply

$$0 < \rho(Z/V) = \langle -\mu, 1_{A_n} \rangle - \rho^*(\mu) \leq \langle -\mu, 1_{A_n} \rangle \quad (10)$$

for some  $\mu \in \mathcal{C}$ . Since, moreover,  $1 < V/U$  on  $A_n$  we infer that  $\langle -\mu, 1_{A_n} \rangle < \langle -\mu, 1_{A_n} V/U \rangle$  and therefore

$$\rho(Z/V) < \langle -\mu, 1_{A_n} V/U \rangle - \rho^*(\mu) = \langle \mu, Z/U \rangle - \rho^*(\mu) \leq \rho(Z/U).$$

But this contradicts the assumption of the theorem, whence  $\mathbb{P}[A_n] = 0$ . By letting  $n \rightarrow \infty$ , we conclude  $U \geq V$ .

This also implies  $V \in \mathcal{M}$  and hence  $\rho(V/U) \leq \rho(V/V) = -1$ . Define  $B := \{U > V\}$ . If  $\mathbb{P}[B] > 0$  then, by (6),

$$0 < \rho(-1 + V/U) = 1 + \rho(V/U) \leq 1 - 1 = 0,$$

a contradiction. Hence  $\mathbb{P}[B] = 0$  and thus  $U = V$ .  $\square$

**Remark 1.2.** Condition (6) is satisfied by many known convex risk measures, such as expected shortfall (see e.g. [6]). Expected shortfall is the underlying risk measure in the Swiss Solvency Test [7], the new regulatory framework for Swiss insurance companies. Moreover, it is internally used by some major insurance companies (see [4]).

**Remark 1.3.** The conclusion of the theorem becomes stronger the smaller the set  $\mathcal{M}$  of “test positions” is. An inspection of the proof shows that it would suffice to consider elements  $Z \in \mathcal{M}$  with  $Z/V \leq \epsilon$ , for some  $\epsilon > 0$ .

**Remark 1.4.** The risk measure considered the theorem,  $\rho_U(Z) := \rho(Z/U)$ , satisfies convexity (1), monotonicity (2) and normality (4). However, cash-invariance (3) has to be replaced by  $U$ -invariance:

$$\rho_U(Z + mU) = \rho_U(Z) - m, \quad \text{for } m \in \mathbb{R}.$$

For a more detailed study of such risk measures see [3].

**Remark 1.5.** Artzner et al. [2] (henceforth ADK) also examine the effect of a change of numeraire on risk measures, albeit in a different context. Indeed, after a slight adaptation of notation, they fix a set  $\mathcal{A}$  of acceptable terminal nominal portfolio values and a pair of numeraires  $U, V > 0$  with initial value one. Let  $\mathcal{M}_0 = \{x(U - V) \mid x \in \mathbb{R}\}$  denote the space of portfolios in  $U$  and  $V$  with zero initial value. In fact, ADK consider more than two tradeable assets, but the minimum set consists of  $U$  and  $V$ . For any terminal nominal value  $X$ , ADK define the minimum additional capital invested in  $U$  and  $V$  at inception for  $X$  to become  $\mathcal{A}$ -acceptable

$$\begin{aligned} \rho_{ADK}(X) &= \inf\{m \mid X + xU + yV \in \mathcal{A}, \text{ for some } x + y = m\} \\ &= \inf\{m \mid X + mU \in \mathcal{A} + \mathcal{M}_0\} = \inf\{m \mid X + mV \in \mathcal{A} + \mathcal{M}_0\}. \end{aligned}$$

Obviously, the risk measure  $\rho_{ADK}$  is both  $U$ - and  $V$ -invariant (see Remark 1.4). In this sense, the augmented acceptance set  $\mathcal{A} + \mathcal{M}_0$  is “numeraire invariant” with respect to  $U$  and  $V$ .

Our approach is different as we started with a fixed convex risk measure  $\rho$ , satisfying axioms (1)–(4). Any choice of a numeraire  $U$  induced a corresponding set of acceptable nominal portfolio values  $\mathcal{A}^U = \{X \mid \rho(X/U) \leq 0\} = U\mathcal{A}^1$ . Our objective was then to find an optimal numeraire, which in particular would maximize the acceptance set  $\mathcal{A}^U$ . This approach is closer to practice, where it is more common to explicitly specify a risk measure (a “simple” object) first, which then implies an acceptance set (a “complex” object), than the other way round.

Finally, let us consider a somewhat related problem: for two convex risk measures  $\rho$  and  $\sigma$  on  $L^\infty$ , does  $\sigma \leq \rho$  imply  $\sigma = \rho$ ? The answer is no. Actually, any subgradient  $\sigma \in \partial\rho(0) := \{\nu \in (L^\infty)^* \mid \langle \nu, Z \rangle \leq \rho(Z) \forall Z \in L^\infty\}$  defines a convex risk measure with this property. Indeed, it is well known (see e.g. [3]) that  $\emptyset \neq \partial\rho(0) \subset \mathcal{C}$ , see (7).

## 2 Conclusion

I have shown that, under a reasonable technical condition, there is no optimal numeraire that yields lower solvency capital requirements than any other numeraire. In particular, the greater than risk free expected return of a risky numeraire cannot compensate for the additional risk that is introduced when discounting by its terminal value.

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