

# Optimal Capital and Risk Allocations for Law- and Cash-Invariant Convex Functions\*

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## Abstract

In this paper we provide the complete solution to the existence and characterisation problem of optimal capital and risk allocations for not necessarily monotone, law-invariant convex risk measures on the model space  $L^p$ , for any  $p \in [1, \infty]$ . Our main result says that the capital and risk allocation problem always admits a solution via contracts whose payoffs are defined as increasing Lipschitz continuous functions of the aggregate risk.

**Key words:** exact convolutions, law-invariant risk measures, optimal capital and risk allocations.

## 1 Introduction

The problem of optimal capital and risk allocation among economic agents, or business units, has played a predominant role in the respective academic and industrial research areas for decades. The introduction of coherent and convex risk measures by Artzner et al. [3], Föllmer and Schied [16], and Frittelli and Rosazza-Gianin [17], respectively, has drawn the attention to study this problem using a new kind of approach (see Barrieu and El Karoui [5], Jouini et al. [18], Filipović and Kupper [11], Burgert and Rüschenendorf [6], Acciaio [1]). For some overview of the vast related finance literature we refer to Dana and Scarsini [8], Burgert and Rüschenendorf [6] and the references therein.

In this paper we provide the complete solution to the existence and characterisation problem of optimal capital and risk allocations for not necessarily

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monotone, law-invariant convex risk measures on the model space  $L^p$ , for any  $p \in [1, \infty]$ . That is, we consider  $n$  agents, or business units, with initial endowments  $X_1, \dots, X_n \in L^p$ , who assess the riskiness of their positions by means of some not necessarily monotone, law-invariant convex risk measures  $\rho_i$  on  $L^p$ . In order to minimise total and individual risk, the agents redistribute the aggregate endowment  $X = X_1 + \dots + X_n$  among themselves. An optimal capital and risk allocation  $Y_1, \dots, Y_n$  satisfies  $Y_1 + \dots + Y_n = X$  and

$$\rho_1(Y_1) + \dots + \rho_n(Y_n) = \inf_{\sum_{i=1}^n Z_i = X} (\rho_1(Z_1) + \dots + \rho_n(Z_n)). \quad (1.1)$$

As is often the case in practice, this redistribution procedure may be subject to frictions (e.g. limited fungibility of capital, see [12, 13]) in the sense that not every allocation of  $X$  is admissible. This can be formalised by restricting the risk measures  $\rho_i$  accordingly, as proposed in [11], see also example 6.1 below. The restricted  $\rho_i$  are typically not monotone, even though the original  $\rho_i$  may be so. But this goes well with our framework, since we do not require monotonicity of  $\rho_i$  right from the start. Examples for  $\rho_i$  are mean-risk type risk measures, such as mean-variance, which obviously are convex law- and cash-invariant, but not monotone on  $L^p$ .

Our main result says that there *always* exists a solution to (1.1) in  $L^p$  of the form  $Y_i = f_i(X)$ , for some increasing Lipschitz continuous functions  $f_i : \mathbb{R} \rightarrow \mathbb{R}$  with  $\sum_{i=1}^n f_i = \text{Id}_{\mathbb{R}}$ . In other words, the capital and risk allocation problem (1.1) always admits a solution via contracts whose payoffs are defined as (increasing Lipschitz continuous) functions of the aggregate risk  $X$ . This extremely useful fact is often assumed in economic contract theory. We now set this prevalent economic assumption on a sound mathematical basis.

As regards the uniqueness of the optimal allocation, one has always the freedom to rebalance the cash (see remark 2.7 below). In this paper, we do not elaborate on general uniqueness results. However, we will provide examples which illustrate that both uniqueness (up to rebalancing the cash) and non-uniqueness can occur.

The existence proof is constructive. Following along the lines of Landsberger and Meilijson [20], we approximate the optimal allocation by simple random variables. At each level of approximation, the respective approximate solution is comonotone and optimal with respect to the approximate aggregate endowment. This allows, in principle, to compute the approximate optimal capital and risk allocation at any given level of accuracy. A useful fact, that will be explored elsewhere.

The article of Jouini et al. [18] has been most influential for this paper. Indeed, Jouini et al. [18] provide existence of optimal allocations for *monotone* law-invariant convex risk measures  $\rho_i$  on  $L^\infty$ . Our motivation was to understand and extend their results beyond  $L^\infty$ , which from an applied perspective is a very limited model space (e.g.  $L^\infty$  does not contain normal distributed random variables). Moreover, in view of the predominant use of mean-variance risk preferences in the literature and also the framework in [11], it was necessary to abandon the monotonicity assumption. While developing our results, we

became aware of the article [1] by Acciaio, where she claims existence of optimal allocations on  $L^\infty$  without assuming monotonicity of  $\rho_i$ . However, after some inquiries, it turned out that she had no proof available. In that sense, our paper can be considered as a cooperative essential completion of [1].

The remainder of the paper is as follows. In section 2, we state our main result (theorem 2.5). In section 3, we reduce the  $n$ -agent problem on  $L^p$  to the case  $n = 2$  and  $p = 1$ . Section 4 contains the core of the proof of theorem 2.5. This is a result of Landsberger and Meilijson [20], which however they only proved for simple random variables. We thus provide a full and comprehensive proof. In section 5 we accomplish the proof of theorem 2.5. Section 6 contains examples which further illustrate the existence and uniqueness properties of optimal allocations. We suppose the reader is familiar with basic duality theory for convex functions as outlined in [21] or [10]. In section A, we give a brief summary of notational conventions and results from convex analysis which will be used throughout the text. Sections B and C contain lemmas that are needed for the proof of our main results.

## 2 Statement of Main Result

Throughout this paper  $(\Omega, \mathcal{F}, \mathbb{P})$  denotes an atom-less probability space, i.e. a probability space supporting a random variable with continuous distribution. All equalities and inequalities between random variables are understood in the  $\mathbb{P}$ -almost sure (a.s.) sense. If not specified otherwise, in the sequel, we let  $p \in [1, \infty]$ , and write  $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$  and  $\|\cdot\|_p = \|\cdot\|_{L^p}$ . The topological dual space of  $L^p$  is denoted by  $(L^p)^*$ . It is well known that  $(L^p)^* = L^q$  with  $q = \frac{p}{p-1}$  for  $p < \infty$ , and that  $(L^\infty)^* \supset L^1$  can be identified with  $ba$ , the space of all bounded finitely additive measures on  $(\Omega, \mathcal{F})$  which are absolutely continuous with respect to  $\mathbb{P}$ . With some facilitating abuse of notation, we shall write  $(X, Z) \mapsto E[XZ]$  for the dual pairing on  $(L^p, (L^p)^*)$  also for the case  $p = \infty$ .

Before we state our main result, let us first define the objects we are studying. We write  $X \sim Y$  if two random variables  $X$  and  $Y$  are identically distributed.

**Definition 2.1.** *A function  $F : L^p \rightarrow (-\infty, \infty]$  is*

- (i) convex if  $F(\lambda X + (1 - \lambda)Y) \leq \lambda F(X) + (1 - \lambda)F(Y)$  for all  $\lambda \in [0, 1]$ ,
- (ii) cash-invariant if  $F(0) < \infty$  and  $F(X + m) = F(X) - m$  for all  $m \in \mathbb{R}$ ,
- (iii) monotone if  $X \geq Y$  implies  $F(X) \leq F(Y)$ ,
- (iv) positively homogeneous if  $F(tX) = tF(X)$  for all  $t \geq 0$ ,
- (v) law-invariant if  $X \sim Y$  implies  $F(X) = F(Y)$ .

**Definition 2.2.** *A convex risk measure on  $L^p$  is a monotone convex cash-invariant function  $L^p \rightarrow (-\infty, \infty]$ . A coherent risk measure is a convex risk measure which in addition is positively homogeneous.*

Let  $\rho : L^p \rightarrow (-\infty, \infty]$  be a convex cash-invariant function. The set of all acceptable positions with respect to  $\rho$  is  $\mathcal{A}_\rho := \{X \in L^p \mid \rho(X) \leq 0\}$ . Note that, by cash-invariance,  $\rho$  is lower semi-continuous (l.s.c.) if and only if its acceptance set  $\mathcal{A}_\rho$  is closed.

We now formalise the capital and risk allocation problem in terms of the (infimal) convolution of convex cash-invariant functions. The convolution operation, being the dual of the summation, is in fact a well-known concept from convex analysis:

**Definition 2.3.** For  $n \geq 2$  and  $n$  functions  $F_1, F_2, \dots, F_n : L^p \rightarrow [-\infty, \infty]$  we define the convolution  $F_1 \square F_2 \square \dots \square F_n = \square_{i=1}^n F_i : L^p \rightarrow [-\infty, \infty]$  of these functions by

$$\square_{i=1}^n F_i(X) = \inf_{\substack{X_1, \dots, X_n \in L^p, \\ \sum_{i=1}^n X_i = X}} \sum_{i=1}^n F_i(X_i).$$

Let  $X \in L^p$ . An  $n$ -tuple  $(X_1, X_2, \dots, X_n) \in L^p \times \dots \times L^p$  such that  $\sum_{i=1}^n X_i = X$  is called an allocation of  $X$ . The convolution  $\square_{i=1}^n F_i$  is said to be exact at  $X$  if there exists an allocation  $(X_1, X_2, \dots, X_n)$  of  $X$  such that  $\square_{i=1}^n F_i(X) = \sum_{i=1}^n F_i(X_i)$ . Such a minimising allocation is called an optimal allocation of  $X$ . The convolution is said to be exact if it is exact at every point  $X \in L^p$ .

Hence, the capital and risk allocation problem outlined in section 1 is equivalent to finding an optimal allocation for the convolution  $\square_{i=1}^n \rho_i$  of  $n$  convex cash-invariant functions  $\rho_i$ .

**Definition 2.4.** An allocation  $(X_1, \dots, X_n)$  of  $X \in L^p$  is called comonotone if there exist increasing functions  $f_1, \dots, f_n : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\sum_{i=1}^n f_i = \text{Id}_{\mathbb{R}}$  and  $X_i = f_i(X)$  for all  $i$ . These functions  $f_i$  are obviously 1-Lipschitz-continuous.

The following theorem is the main result of this paper. The proof is given in section 5.

**Theorem 2.5.** Let  $\rho_1, \dots, \rho_n : L^p \rightarrow (-\infty, \infty]$  be l.s.c. law-invariant convex cash-invariant functions. Then  $\square_{i=1}^n \rho_i$  is a l.s.c. law-invariant convex cash-invariant function on  $L^p$ . Moreover, for every  $X \in L^p$  there exists a comonotone allocation  $(X_1, \dots, X_n)$  such that

$$\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i). \quad (2.2)$$

In other words,  $\square_{i=1}^n \rho_i$  is exact, and amongst the optimal allocations of any  $X \in L^p$  there is always a comonotone one.

**Remark 2.6.** The economic message of theorem 2.5 is that the capital and risk allocation problem (1.1) always admits a solution via contracts whose payoffs are defined as (increasing Lipschitz continuous) functions  $f_i(X)$  of the aggregate risk  $X$ . We note that this extremely useful fact is often *assumed* in economic contract theory. Theorem 2.5 now sets this prevalent economic assumption on a sound mathematical basis. ||

**Remark 2.7.** The required l.s.c. in theorem 2.5 cannot be dropped (see example 6.5). As regards the uniqueness of the optimal allocation, clearly, due to cash-invariance of  $\rho_i$ , we can only expect uniqueness up to *rebalancing the cash*. That is,  $(X_1, \dots, X_n)$  is an optimal allocation of  $X$  if and only if  $(X_1 + c_1, \dots, X_n + c_n)$  is so, for all cash positions  $c_i \in \mathbb{R}$  with  $\sum_{i=1}^n c_i = 0$ . In this paper we do not elaborate on general uniqueness results. However, example 6.6 below illustrates that even in the class of comonotone allocations, the optimal may not be unique (up to rebalancing the cash). On the other hand, example 6.7 below shows that there are situations where the optimal allocation is unique (up to rebalancing the cash).  $\parallel$

We note that the functions  $\rho_i$  in theorem 2.5 do not have to be monotone. In case at least one of them is monotone (i.e. a convex risk measure), we may draw the following stronger conclusion:

**Corollary 2.8.** *Let  $\rho_1, \dots, \rho_n : L^p \rightarrow (-\infty, \infty]$  be l.s.c. law-invariant convex cash-invariant functions, of which at least one is a convex risk measure. Then,  $\square_{i=1}^n \rho_i$  is a l.s.c. law-invariant convex risk measure on  $L^p$ . Moreover, for every  $X \in L^p$  there exists a comonotone optimal allocation.*

*Proof.* In view of theorem 2.5 it remains to prove that  $\square_{i=1}^n \rho_i$  is monotone. But this follows immediately from lemma 3.1 below and the fact that a l.s.c. proper convex function  $F : L^p \rightarrow (-\infty, \infty]$  is monotone if and only if  $\text{dom } F^* \subset (L^p)^*_-$  (see e.g. [14] lemma 3.2).  $\square$

### 3 Preliminaries and Problem Reduction

The convolution defined in definition 2.3 is commutative and associative. Hence, it suffices to reduce our examinations of this operation to the case when  $n = 2$ . The general case follows inductively. Moreover, it is easily verified that the convolution preserves convexity in the sense that if both  $F_1$  and  $F_2$  are convex, then so is  $F_1 \square F_2$ . Other well-known, and in the sequel used, facts are:

**Lemma 3.1.** (i)  $\text{dom } F_1 \square F_2 = \text{dom } F_1 + \text{dom } F_2$ ,

(ii)  $(F_1 \square F_2)^* = F_1^* + F_2^*$ ,

(iii)  $\text{dom } (F_1 \square F_2)^* = \text{dom } F_1^* \cap \text{dom } F_2^*$ .

*Proof.* (i) is obvious. (ii) is due to:

$$\begin{aligned} (F_1 \square F_2)^*(Z) &= \sup_X E[XZ] - \inf_{X_1 + X_2 = X} F_1(X_1) + F_2(X_2) \\ &= \sup_X \sup_{X_1 + X_2 = X} E[X_1 Z] + E[X_2 Z] - F_1(X_1) - F_2(X_2) \\ &= \sup_{X_1, X_2} E[X_1 Z] + E[X_2 Z] - F_1(X_1) - F_2(X_2) \\ &= F_1^*(Z) + F_2^*(Z), \end{aligned}$$

and (iii) is an immediate consequence of (ii).  $\square$

Let

$$\mathbb{A} := \{(f, g) \mid f, g : \mathbb{R} \rightarrow \mathbb{R} \text{ are increasing, } f + g = \text{Id}_{\mathbb{R}}\}.$$

Clearly, if  $(f, g) \in \mathbb{A}$ , then both  $f$  and  $g$  are 1-Lipschitz-continuous. Hence,  $|f(X)| \leq |X| + |f(0)|$  and  $|g(X)| \leq |X| + |g(0)|$ , implying that

$$\text{if } X \in L^p \text{ then } (f(X), g(X)) \in L^p \times L^p. \quad (3.3)$$

Thus, for any  $X \in L^p$  the set  $\{(f(X), g(X)) \mid (f, g) \in \mathbb{A}\}$  is the set of all comonotone 2-dimensional allocations of  $X$ .

For the sake of simplicity, we will further restrict our studies to the case  $p = 1$ . By nature of the arguments presented in the proof of theorem 2.5 (section 5), it will become clear that they all literally carry over to  $L^p$ , simply by replacing  $L^1$  with  $L^p$  and choosing the appropriate dual. However, in what follows, we give another justification for the retreat to  $L^1$  by proving that the assertions of theorem 2.5 for the  $L^p$ -cases can be derived as a corollary from knowing it for the case of  $L^1$ . The main ingredient to this is the following result:

**Theorem 3.2.** *Let  $\rho : L^p \rightarrow (-\infty, \infty]$  be a l.s.c. law-invariant convex cash-invariant function. Then, there is a unique l.s.c. law-invariant convex cash-invariant function  $\tilde{\rho} : L^1 \rightarrow (-\infty, \infty]$  such that  $\rho = \tilde{\rho}|_{L^p}$ .*

*Proof.* see [15] theorem 3.3 and remark 3.5. □

**Corollary 3.3.** *Let  $\rho_1, \rho_2 : L^p \rightarrow (-\infty, \infty]$  be two l.s.c. law-invariant convex cash-invariant functions, and let  $\tilde{\rho}_1$  and  $\tilde{\rho}_2$  be the unique l.s.c. law-invariant convex cash-invariant functions on  $L^1$  such that  $\rho_i = \tilde{\rho}_i|_{L^p}$ ,  $i = 1, 2$ . Suppose the assertions of theorem 2.5 hold for  $\tilde{\rho}_1 \square \tilde{\rho}_2$ . Then  $\rho_1 \square \rho_2 = \tilde{\rho}_1 \square \tilde{\rho}_2|_{L^p}$ . In particular, the assertions of theorem 2.5 are true for  $\rho_1 \square \rho_2$  too.*

*Proof.* By assumption, for any  $X \in L^p \subset L^1$  there is a comonotone allocation  $(f(X), g(X))$  such that  $\tilde{\rho}_1 \square \tilde{\rho}_2(X) = \tilde{\rho}_1(f(X)) + \tilde{\rho}_2(g(X))$ . Clearly,  $\tilde{\rho}_1 \square \tilde{\rho}_2(X) \leq \rho_1 \square \rho_2(X)$ , and since  $(f(X), g(X)) \in L^p \times L^p$  by (3.3), we deduce that

$$\tilde{\rho}_1 \square \tilde{\rho}_2(X) = \rho_1(f(X)) + \rho_2(g(X)) = \rho_1 \square \rho_2(X).$$

Hence, firstly,  $\rho_1 \square \rho_2$  is simply the restriction of  $\tilde{\rho}_1 \square \tilde{\rho}_2$  to  $L^p$  and thus a l.s.c. (w.r.t.  $\|\cdot\|_p$ ) law-invariant convex cash-invariant function. (The l.s.c. stems from the fact that  $\|\cdot\|_p$ -convergence implies  $\|\cdot\|_1$ -convergence.) Secondly,  $\rho_1 \square \rho_2$  is exact and there is always a comonotone optimal allocation. □

## 4 Comonotone Concave Order Improvement

We denote by  $\succeq_c$  the concave order on  $L^1$ , that is,  $X \succeq_c Y$  if and only if  $E[u(X)] \geq E[u(Y)]$  for all concave functions  $u : \mathbb{R} \rightarrow \mathbb{R}$ . Clearly, since  $\text{Id}_{\mathbb{R}}$  and  $-\text{Id}_{\mathbb{R}}$  are concave functions,  $X \succeq_c Y$  implies  $E[X] = E[Y]$ . Moreover,

$$X \succeq_c Y \quad \Leftrightarrow \quad E[X] = E[Y] \text{ and } E[(X - c)^+] \leq E[(Y - c)^+] \quad \forall c \in \mathbb{R}. \quad (4.4)$$

For a proof of (4.4), we refer to corollary 2.62 in [16].

Proposition 4.1 below will turn out to be the basis of the proof of theorem 2.5. It is based upon the results of Landsberger and Meilijson [20], and states that every allocation is dominated in concave order by a comonotone allocation. The importance of this results becomes clear by (B.13) where we establish that any l.s.c. law-invariant convex function is monotone w.r.t. the  $\succeq_c$ -order.

**Proposition 4.1.** *(see proposition 1 in [20]) For any allocation  $(Y, Z)$  of  $X \in L^1$ , there is  $(f, g) \in \mathbb{A}$  such that  $f(X) \succeq_c Y$  and  $g(X) \succeq_c Z$ .*

Unfortunately, Landsberger and Meilijson [20] only proved this result for random variables  $X$  supported by a finite set. For sake of completeness, we thus give a full proof here.

*Proof.* We divide the proof into three steps.

**Step 1:** We start out as in [20] by noticing that Jensen's inequality implies that  $(E[Y | X], E[Z | X])$  is an allocation of  $X$  which is at least as good as  $(Y, Z)$ , meaning that  $E[Y | X] \succeq_c Y$  and  $E[Z | X] \succeq_c Z$ . Let  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  be measurable functions such that  $h_1(X) = E[Y | X]$ ,  $h_2(X) = E[Z | X]$ . Clearly, we may assume that  $h_1 + h_2 = \text{Id}_{\mathbb{R}}$ . If  $h_1$  and  $h_2$  are increasing, we are done, if not, we go on improving this allocation. However, we have now established that during the remainder of this proof we may restrict ourselves to improve allocations  $(Y, Z)$  of type  $Y = h_1(X)$  and  $Z = h_2(X)$  for some measurable functions  $h_1, h_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $h_1 + h_2 = \text{Id}_{\mathbb{R}}$ .

**Step 2:** Suppose  $X$  is a simple random variable, i.e.  $X = \sum_{i=1}^n x_i 1_{A_i}$  for a partition  $A_1, \dots, A_n$  of  $\Omega$  and real numbers  $x_i$  such that  $x_i \neq x_j$  for  $i \neq j$ . Let  $y_i := h_1(x_i)$  and  $z_i := h_2(x_i)$ . Then  $h_1(X) = \sum_{i=1}^n y_i 1_{A_i}$  and  $h_2(X) = \sum_{i=1}^n z_i 1_{A_i}$ . We set  $x := (x_1, \dots, x_n)$ ,  $y := (y_1, \dots, y_n)$ ,  $z := (z_1, \dots, z_n)$  and  $p_k := \mathbb{P}(A_k)$ ,  $k = 1, \dots, n$ . Let  $\pi$  be a permutation of  $\{1, \dots, n\}$  such that  $x_\pi := (x_{\pi(1)}, \dots, x_{\pi(n)}) \in \mathcal{D} := \{\tilde{x} \in \mathbb{R}^n \mid \tilde{x}_1 \leq \tilde{x}_2 \leq \dots \leq \tilde{x}_n\}$ . Observe that  $(h_1(X), h_2(X))$  is comonotone if and only if  $y_\pi, z_\pi \in \mathcal{D}$ . For sake of brevity we may and will assume w.l.o.g. that  $x \in \mathcal{D}$  already. Supposing that  $(y, z)$  is not comonotone, i.e.  $y \notin \mathcal{D}$  or  $z \notin \mathcal{D}$  or both, the following algorithm by M. Landsberger and I. Meilijson transfers  $(y, z)$  into a comonotone allocation:

Since  $(y, z)$  is not comonotone, there must exist an  $i$  such that  $y_1 \leq \dots \leq y_i$ ,  $z_1 \leq \dots \leq z_i$  but either  $y_{i+1} < y_i$  or  $z_{i+1} < z_i$ . W.l.o.g. let us assume that  $z_{i+1} < z_i$ . Then there is a smallest  $j$  such that  $z_{i+1} < z_j$ . For  $k = j, \dots, i$  we set

$$y_k^{new} = y_k + \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l} (z_j - z_{i+1}) \quad \text{and} \quad z_k^{new} = z_k - \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l} (z_j - z_{i+1})$$

whereas

$$y_{i+1}^{new} = y_{i+1} - \frac{\sum_{l=j}^i p_l}{\sum_{l=j}^{i+1} p_l} (z_j - z_{i+1}) \quad \text{and} \quad z_{i+1}^{new} = z_{i+1} + \frac{\sum_{l=j}^i p_l}{\sum_{l=j}^{i+1} p_l} (z_j - z_{i+1}).$$

The other coordinates of  $y$  and  $z$  are left unchanged. Finally, set  $y := y^{new}$  and  $z := z^{new}$  and repeat the procedure in case the output is not comonotone.

Let  $(Y^{new}, Z^{new}) := (\sum_{i=1}^n y_i^{new} 1_{A_i}, \sum_{i=1}^n z_i^{new} 1_{A_i})$ . Firstly,  $(Y^{new}, Z^{new})$  is obviously an allocation of  $X$ , secondly, we claim that  $Y^{new} \succeq_c Y$  and  $Z^{new} \succeq_c Z$ , i.e. each cycle of the algorithm improves the allocation, and finally, it is easily verified that the algorithm returns a comonotone allocation in at most  $n(n-1)/2$  cycles (observe that  $z_j^{new} = z_{i+1}^{new}$ ). In order to show that  $Y^{new} \succeq_c Y$  and  $Z^{new} \succeq_c Z$ , let  $u : \mathbb{R} \rightarrow \mathbb{R}$  be any concave function. Introducing the abbreviations

$$\alpha := \frac{p_{i+1}}{\sum_{l=j}^{i+1} p_l} \in (0, 1) \quad \text{and} \quad \lambda_k := \frac{z_j - z_{i+1}}{z_k - z_{i+1}} \in (0, 1]$$

and recalling that concavity is equivalent to

$$\forall a < b < c : \frac{u(b) - u(a)}{b - a} \geq \frac{u(c) - u(a)}{c - a} \geq \frac{u(c) - u(b)}{c - b}, \quad (4.5)$$

we compute:

$$\begin{aligned} \sum_{k=j}^{i+1} u(z_k^{new}) p_k &= \sum_{k=j}^i u((1 - \alpha \lambda_k) z_k + \alpha \lambda_k z_{i+1}) p_k + u((1 - \alpha) z_j + \alpha z_{i+1}) p_{i+1} \\ &\geq \sum_{k=j}^i [(1 - \alpha \lambda_k) u(z_k) + \alpha \lambda_k u(z_{i+1})] p_k \\ &\quad + [(1 - \alpha) u(z_j) + \alpha u(z_{i+1})] p_{i+1} \\ &= \sum_{k=j}^{i+1} u(z_k) p_k + (1 - \alpha) (u(z_j) - u(z_{i+1})) p_{i+1} \\ &\quad - \alpha \sum_{k=j}^i \lambda_k (u(z_k) - u(z_{i+1})) p_k \\ &\stackrel{(4.5)}{\geq} \sum_{k=j}^{i+1} u(z_k) p_k, \end{aligned}$$

because  $\lambda_k (u(z_k) - u(z_{i+1})) \leq u(z_j) - u(z_{i+1})$  by inequality (4.5). A similar computation for  $Y^{new}$  shows that  $Y^{new} \succeq_c Y$  and  $Z^{new} \succeq_c Z$ .

**Step 3:** Let  $X$  be any integrable random variable. Recalling the usual monotone approximation from Lebesgue integration theory, let  $(Y_n)_{n \in \mathbb{N}}$  and  $(Z_n)_{n \in \mathbb{N}}$  be sequences of simple random variables converging  $\mathbb{P}$ -a.s. and in  $L^1$  to  $Y$  and  $Z$  respectively such that  $|Y_n| \leq |Y|$  and  $|Z_n| \leq |Z|$  for all  $n \in \mathbb{N}$ . Then  $X_n := Y_n + Z_n$  converges to  $X$   $\mathbb{P}$ -a.s. and in  $L^1$ . By step 2, for each  $n \in \mathbb{N}$ , there exists a comonotone improvement  $(f_n(X_n), g_n(X_n))$  of  $(Y_n, Z_n)$ . Choose  $N \in \mathbb{N}$  such that  $\|Y_n\|_1 \leq \|Y\|_1 + 1$ ,  $\|Z_n\|_1 \leq \|Z\|_1 + 1$ , and  $\|X_n\|_1 \leq \|X\|_1 + 1$  for all  $n \geq N$ . Since all  $f_n$  (and  $g_n$ ) are 1-Lipschitz-continuous, we have that

$|f_n(0)| \leq |X_n| + |f_n(X_n)|$ . Taking expectations on both sides yields

$$|f_n(0)| \leq E[|X_n|] + E[|f_n(X_n)|] \leq E[|X_n|] + E[|Y_n|],$$

because  $f_n(X_n) \succeq_c Y_n$  and  $x \mapsto -|x|$  is concave. Hence, if  $n \geq N$ , we get  $|f_n(0)| \leq \|X\|_1 + \|Y\|_1 + 2 =: K_1$  and similarly  $|g_n(0)| \leq \|X\|_1 + \|Z\|_1 + 2 =: K_2$ , and thus  $f_n(0), g_n(0) \in [-K, K]$  for  $K := \max\{K_1, K_2\}$ . Therefore, by lemma C.1, there is a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  and a 1-Lipschitz-continuous increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(a) = \lim_{k \rightarrow \infty} f_{n_k}(a)$ ,  $a \in \mathbb{R}$ . Now it is easily verified that  $(g_{n_k})_{k \in \mathbb{N}}$  converges pointwise to the 1-Lipschitz-continuous increasing function  $g := \text{Id}_{\mathbb{R}} - f$ . Hence, the sequence  $f_{n_k}(X_{n_k})$  converges  $\mathbb{P}$ -a.s. to  $f(X)$ , and  $g_{n_k}(X_{n_k}) = X_{n_k} - f_{n_k}(X_{n_k})$  converges  $\mathbb{P}$ -a.s. to  $g(X)$ . Since  $|f_{n_k}(X_{n_k})| \leq |X_{n_k}| + K \leq |Y| + |Z| + K$  and  $|g_{n_k}(X_{n_k})| \leq |Y| + |Z| + K$  for large enough  $k \in \mathbb{N}$ , we can apply the dominated convergence theorem which yields  $f(X), g(X) \in L^1$  and  $\|f(X) - f_{n_k}(X_{n_k})\|_1 \rightarrow 0$ ,  $\|g(X) - g_{n_k}(X_{n_k})\|_1 \rightarrow 0$  for  $k \rightarrow \infty$ . Moreover, in view of (4.4), we have that

$$E[f(X)] = \lim_{k \rightarrow \infty} E[f_{n_k}(X_{n_k})] = \lim_{k \rightarrow \infty} E[Y_{n_k}] = E[Y],$$

and for all  $c \in \mathbb{R}$ :

$$\begin{aligned} E[(f(X) - c)^+] &= \lim_{k \rightarrow \infty} E[(f_{n_k}(X_{n_k}) - c)^+] \\ &\leq \lim_{k \rightarrow \infty} E[(Y_{n_k} - c)^+] = E[(Y - c)^+], \end{aligned}$$

and similarly for  $g$ . Hence,  $(f(X), g(X))$  is a comonotone allocation of  $X$  satisfying  $f(X) \succeq_c Y$  and  $g(X) \succeq_c Z$  according to (4.4).  $\square$

## 5 Proof of Theorem 2.5

In view of corollary 3.3 we only have to prove theorem 2.5 for  $p = 1$ . To this end, let  $\rho_1, \rho_2 : L^1 \rightarrow (-\infty, \infty]$  be l.s.c. law-invariant convex cash-invariant functions. We divide the proof into four steps.

**Step 1:**  $\rho_1 \square \rho_2$  is proper, convex, and cash-invariant.

*Proof.* It is easily verified that the convolution preserves the convexity of  $\rho_1$  and  $\rho_2$ . By (B.15) we have that  $\rho_i(X) \geq -E[X] + \rho_i(0)$  for all  $X \in L^1$  and  $i = 1, 2$ . Hence,

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \inf_{X_1 + X_2 = X} \rho_1(X_1) + \rho_2(X_2) \\ &\geq \inf_{X_1 + X_2 = X} -E[X_1] - E[X_2] + \rho_1(0) + \rho_2(0) \\ &= -E[X] + \rho_1(0) + \rho_2(0), \end{aligned}$$

so  $\rho_1 \square \rho_2$  is proper and  $\rho_1 \square \rho_2(0) = \rho_1(0) + \rho_2(0) < \infty$ . Furthermore, for all  $r \in \mathbb{R}$  we get

$$\rho_1 \square \rho_2(X + r) = \inf_{Y \in L^1} \rho_1(X + r - Y) + \rho_2(Y) = \rho_1 \square \rho_2(X) - r$$

due to the cash-invariance of  $\rho_1$ .  $\square$

**Step 2:**  $\rho_1 \square \rho_2(X) = \inf_{(f,g) \in \mathbb{A}} \rho_1(f(X)) + \rho_2(g(X))$ ,  $X \in L^1$ .

*Proof.* This is an immediate consequence of proposition 4.1 and (B.13).  $\square$

**Step 3:**  $\rho_1 \square \rho_2$  is exact, and for each  $X \in L^1$  there exists a comonotone optimal allocation.

*Proof.* Suppose  $X \in L^1$  is such that  $\rho_1 \square \rho_2(X) = \infty$ . Then every allocation  $(f(X), g(X))$ ,  $(f, g) \in \mathbb{A}$ , is optimal.

Now let  $X \in \text{dom } \rho_1 \square \rho_2$  and choose a sequence  $(f_n, g_n) \in \mathbb{A}$ ,  $n \in \mathbb{N}$ , such that  $\rho_1 \square \rho_2(X) = \lim_{n \rightarrow \infty} \rho_1(f_n(X)) + \rho_2(g_n(X))$ . By cash-invariance we may assume that  $f_n(0) = g_n(0) = 0$  for all  $n \in \mathbb{N}$ . Hence, by lemma C.1, there is a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  and a 1-Lipschitz-continuous and increasing function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(a) = \lim_{k \rightarrow \infty} f_{n_k}(a)$  for all  $a \in \mathbb{R}$ . Clearly, the sequence  $f_{n_k}(X)$  converges  $\mathbb{P}$ -a.s. to  $f(X)$  and  $g_{n_k}(X) = X - f_{n_k}(X)$  converges  $\mathbb{P}$ -a.s. to  $g(X)$  where  $g := \text{Id}_{\mathbb{R}} - f$  is a 1-Lipschitz-continuous increasing function. Since  $|f_{n_k}(X)| \leq |X|$  and  $|g_{n_k}(X)| \leq |X|$  for all  $k \in \mathbb{N}$ , we may apply the dominated convergence theorem which yields  $f(X), g(X) \in L^1$  and  $\|f(X) - f_{n_k}(X)\|_1 \rightarrow 0$ ,  $\|g(X) - g_{n_k}(X)\|_1 \rightarrow 0$  for  $k \rightarrow \infty$ . On the one hand, by l.s.c., we have

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \lim_{k \rightarrow \infty} \rho_1(f_{n_k}(X)) + \rho_2(g_{n_k}(X)) \\ &\geq \liminf_{k \rightarrow \infty} \rho_1(f_{n_k}(X)) + \liminf_{k \rightarrow \infty} \rho_2(g_{n_k}(X)) \\ &\stackrel{\text{l.s.c.}}{\geq} \rho_1(f(X)) + \rho_2(g(X)). \end{aligned}$$

On the other hand, merely by definition of the convolution, we know that  $\rho_1 \square \rho_2(X) \leq \rho_1(f(X)) + \rho_2(g(X))$ . Consequently, the comonotone allocation  $(f(X), g(X))$  of  $X$  is optimal.  $\square$

**Step 4:**  $\rho_1 \square \rho_2$  is l.s.c. and law-invariant.

*Proof.* We claim that  $\mathcal{A}_{\rho_1 \square \rho_2}$  is closed. To this end, let  $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}_{\rho_1 \square \rho_2}$  be a sequence converging to some  $X$  in  $L^1$ . According to step 3 there are  $(f_n, g_n) \in \mathbb{A}$ ,  $n \in \mathbb{N}$ , such that  $0 \geq \rho_1 \square \rho_2(X_n) = \rho_1(f_n(X_n)) + \rho_2(g_n(X_n))$ . By cash-invariance we may assume that  $f_n(0) = g_n(0) = 0$  for all  $n \in \mathbb{N}$ . Similar to step 3, employing lemma C.1, we find a subsequence  $(f_{n_k}, g_{n_k})_{k \in \mathbb{N}}$  of  $(f_n, g_n)_{n \in \mathbb{N}}$  and  $(f, g) \in \mathbb{A}$  such that  $f_{n_k}(X_{n_k})$  converges to  $f(X)$  in  $L^1$  and  $g_{n_k}(X_{n_k})$  converges to  $g(X)$  in  $L^1$ . By l.s.c. of  $\rho_1$  and  $\rho_2$  we get

$$\begin{aligned} \rho_1 \square \rho_2(X) &\leq \rho_1(f(X)) + \rho_2(g(X)) \\ &\leq \liminf_{k \rightarrow \infty} \rho_1(f_{n_k}(X_{n_k})) + \liminf_{k \rightarrow \infty} \rho_2(g_{n_k}(X_{n_k})) \\ &\leq \liminf_{k \rightarrow \infty} \rho_1(f_{n_k}(X_{n_k})) + \rho_2(g_{n_k}(X_{n_k})) \leq 0, \end{aligned}$$

and thus  $X \in \mathcal{A}_{\rho_1 \square \rho_2}$ . Hence,  $\mathcal{A}_{\rho_1 \square \rho_2}$  is closed, i.e.  $\rho_1 \square \rho_2$  is l.s.c.. The law-invariance of  $\rho_1 \square \rho_2$  stems from  $(\rho_1 \square \rho_2)^* = \rho_1^* + \rho_2^*$  (lemma 3.1) and the fact that a l.s.c. convex function on  $L^1$  is law-invariant if and only if its dual is (see (B.12)).  $\square$

## 6 Examples

The first example is motivated and explained in [11].

**Example 6.1.** Two agents with initial endowments  $X_1$  and  $X_2$  in  $L^p$ , assess their individual risk by means of l.s.c. law-invariant convex risk measures  $\rho_1$  and  $\rho_2$  on  $L^p$ , respectively. In order to minimise total and individual risk, they redistribute the aggregate endowment  $X = X_1 + X_2$  amongst themselves. As is often the case in practice, this redistribution procedure might be subject to some restrictions in the sense that not every risk sharing of  $X$  is admissible. We formalise this by defining the set of admissible risk sharings of  $X$  as

$$A_X := \{(Y_1, Y_2) \in M_1 \times M_2 \mid Y_1 + Y_2 \leq X\}$$

where  $M_i \subset L^p$  are closed convex law-invariant cash-invariant (that is,  $Y \in M_i$  implies  $Y + a \in M_i$  for all  $a \in \mathbb{R}$ ) sets such that  $X_i \in M_i$ ,  $i = 1, 2$ . Note that we allow for “free disposal”, i.e.  $X - Y_1 - Y_2 \geq 0$  for all  $(Y_1, Y_2) \in A_X$ . The optimal risk sharing problem is

$$\inf_{(Y_1, Y_2) \in A_X} \rho_1(Y_1) + \rho_2(Y_2). \quad (6.6)$$

In order to solve (6.6), denote  $\rho_i^{M_i} := \rho_i + \delta(\cdot \mid M_i)$ ,  $i = 1, 2$ . Then

$$\begin{aligned} (6.6) &= \inf_{Y_1, Y_2 \in L^p, Y_1 + Y_2 \leq X} \rho_1^{M_1}(Y_1) + \rho_2^{M_2}(Y_2) \\ &= \rho_1^{M_1} \square \rho_2^{M_2} \square \delta(\cdot \mid L_+^p)(X) \\ &= \rho_1^{M_1} \square \rho_2^{M_2} \square -\text{essinf}(X). \end{aligned}$$

Note that  $\delta(\cdot \mid M_i)$  is proper, l.s.c., law-invariant, and convex. By (B.14) we know that  $\delta(E[Y] \mid M_i) \leq \delta(Y \mid M_i)$  for all  $Y \in L^p$ . Hence  $Y \in M_i$  implies  $E[Y] \in M_i$ , and thus  $\mathbb{R} \subset M_i$  by cash-invariance. We conclude that  $\rho_1^{M_1}$  and  $\rho_2^{M_2}$  are l.s.c. law-invariant convex cash-invariant functions. Since  $-\text{essinf}$  is a l.s.c. law-invariant coherent risk measure, we know by corollary 2.8 that  $\rho_1^{M_1} \square \rho_2^{M_2} \square -\text{essinf}$  is a l.s.c. law-invariant convex risk measure, and that this convolution admits a comonotone optimal allocation  $(Y_1, Y_2, Y_3)$  of  $X$ . If  $\rho_1^{M_1}(Y_1) + \rho_2^{M_2}(Y_2) - \text{essinf}(Y_3) = \infty$ , then any admissible risk sharing of  $X$  is optimal. Otherwise, if  $\rho_1^{M_1}(Y_1) + \rho_2^{M_2}(Y_2) - \text{essinf}(Y_3) < \infty$ , then we have that  $-\text{essinf}(Y_3) < \infty$ , and thus  $(Y_1 + \text{essinf}(Y_3), Y_2)$  is a (Pareto optimal) solution of (6.6). For more background on equilibrium and Pareto optimality in this context, we refer to [11]. We thus have established the existence of equilibria for the law-invariant framework, a substantial improvement of corollary 3.7 in [11]. Note also that  $(Y_1 + \text{essinf}(Y_3), Y_2) = (f(X), g(X))$  for some increasing functions  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ .  $\parallel$

As simple applications of theorem 2.5, in the following, we study the convolution of entropic risk measures (example 6.2) and Average Value at Risks (example 6.4) at different levels, respectively. These convolutions are discussed thoroughly in e.g. [4] or [19] on  $L^\infty$ . In contrast, we provide our results on  $L^1$ . For a detailed account of convex risk measures on  $L^p$ ,  $p \in [1, \infty)$ , we refer to [15].

**Example 6.2.** The convolution of entropic risk measures. The entropic risk measure with parameter  $\beta > 0$  is

$$\text{Entr}_\beta(X) = \frac{1}{\beta} \log E[e^{-\beta X}] = \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_{\mathbb{Q}}[-X] - \frac{1}{\beta} H(\mathbb{Q} | \mathbb{P}), \quad X \in L^1,$$

where  $H(\mathbb{Q} | \mathbb{P}) = E_{\mathbb{Q}}[\log \frac{d\mathbb{Q}}{d\mathbb{P}}]$  denotes the relative entropy and  $\mathcal{M}(\mathbb{P})$  is the set of all probability measures  $\mathbb{Q}$  on  $(\Omega, \mathcal{F})$  such that  $\mathbb{Q} \ll \mathbb{P}$  and  $d\mathbb{Q}/d\mathbb{P}$  is bounded. Let  $0 < \beta \leq \gamma$ . Theorem 2.5 and lemma 3.1 justify the following dual approach, for any  $X \in L^1$ :

$$\begin{aligned} \text{Entr}_\beta \square \text{Entr}_\gamma(X) &= \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_{\mathbb{Q}}[-X] - \frac{1}{\beta} H(\mathbb{Q} | \mathbb{P}) - \frac{1}{\gamma} H(\mathbb{Q} | \mathbb{P}) \\ &= \sup_{\mathbb{Q} \in \mathcal{M}(\mathbb{P})} E_{\mathbb{Q}}[-X] - \frac{\beta + \gamma}{\beta\gamma} H(\mathbb{Q} | \mathbb{P}) \\ &= \text{Entr}_{\frac{\beta\gamma}{\beta+\gamma}}(X). \end{aligned}$$

Now, in the search for comonotone optimal allocations, the following ansatz seems natural. We guess that for any  $X \in L^1$  there must be an (obviously comonotone) optimal allocation amongst the allocations of type  $(aX, bX)$  where  $a \in [0, 1]$  and  $b := 1 - a$ . If so, then

$$\frac{\beta + \gamma}{\beta\gamma} \log E[e^{-\frac{\beta\gamma}{\beta+\gamma} X}] = \frac{1}{\beta} \log E[e^{-\beta a X}] + \frac{1}{\gamma} \log E[e^{-\gamma b X}]$$

which is equivalent to

$$\log E[e^{-\frac{\beta\gamma}{\beta+\gamma} X}] = \frac{\gamma}{\beta + \gamma} \log E[e^{-\beta a X}] + \frac{\beta}{\beta + \gamma} \log E[e^{-\gamma b X}].$$

Clearly,  $a = \frac{\gamma}{\beta+\gamma}$  and  $b = \frac{\beta}{\beta+\gamma}$  satisfy this equation. Hence,  $(\frac{\gamma}{\beta+\gamma} X, \frac{\beta}{\beta+\gamma} X)$  is a comonotone optimal allocation of  $X$ .  $\parallel$

Let  $X \in L^1$ , and denote by  $q_X$  the (left-continuous) quantile function of  $X$ , that is,

$$q_X : (0, 1) \rightarrow \mathbb{R}, \quad q_X(s) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq s\}.$$

The Average Value at Risk at level  $\alpha \in (0, 1]$  of  $X$  is then given by

$$\text{AVaR}_\alpha(X) := \sup \left\{ E_{\mathbb{Q}}[-X] \mid \mathbb{Q} \ll \mathbb{P}, \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha} \right\} = \frac{-1}{\alpha} \int_0^\alpha q_X(s) ds, \quad (6.7)$$

see e.g. [15]. Note that

$$\text{AVaR}_1(X) = E[-X] \quad \text{and} \quad \text{AVaR}_0(X) := \lim_{\alpha \downarrow 0} \text{AVaR}_\alpha(X) = -\text{essinf}(X).$$

**Lemma 6.3.** *Let  $0 \leq \beta < \gamma \leq 1$ . Then*

$$\text{AVaR}_\beta(X) \geq \text{AVaR}_\gamma(X),$$

*and equality holds if and only if  $X \geq c$  a.s. and  $\mathbb{P}[X = c] \geq \gamma$  for some constant  $c \in \mathbb{R}$ . In particular,  $\text{AVaR}_\beta(X) = E[-X]$  if and only if  $X$  is constant.*

*Proof.* The case  $\beta = 0$  is obvious. Suppose  $\beta > 0$ . Since  $q_X$  is increasing, we obviously have

$$(\gamma - \beta) \int_0^\beta q_X(s) ds \leq \beta \int_\beta^\gamma q_X(s) ds,$$

with equality if and only if  $q_X(s) = q_X(\gamma)$  for all  $s \leq \gamma$ . In view of (6.7) this proves the claim.  $\square$

**Example 6.4.** The convolution of Average Value at Risks. Let  $0 \leq \beta \leq \gamma \leq 1$ , then

$$\text{AVaR}_\beta \square \text{AVaR}_\gamma = \text{AVaR}_\gamma. \quad (6.8)$$

This is easily verified employing theorem 2.5, lemma 3.1, and the fact that

$$\text{dom AVaR}_\gamma^* \subset \text{dom AVaR}_\beta^*, \quad (6.9)$$

which is implied by (6.7).  $\parallel$

The following example illustrates that the required l.s.c. in theorem 2.5 cannot be dropped, see remark 2.7.

**Example 6.5.** Let  $\rho_1(X) = -E[X] + \delta(X^- | L^\infty)$ , and  $\rho_2 = \text{AVaR}_\alpha$  on  $L^1$ , for some  $\alpha \in (0, 1)$ . Clearly, the acceptance set  $\mathcal{A}_{\rho_1}$  of  $\rho_1$  is not closed and thus  $\rho_1$  is not l.s.c. on  $L^1$ . We claim that

$$\rho_1 \square \rho_2 = -E. \quad (6.10)$$

Indeed, on the one hand, we know that  $\rho_1^* = \delta(\cdot | \{-1\})$  and that  $\rho_2^*(-1) = 0$  (see (B.16)). Hence,  $(\rho_1 \square \rho_2)^* = \rho_1^* + \rho_2^* = \delta(\cdot | \{-1\})$  which implies  $\rho_1 \square \rho_2 \geq -E$ . On the other hand,

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \inf_{X_1 + X_2 = X} E[-X_1] + \delta(X_1^- | L^\infty) + \text{AVaR}_\alpha(X_2) \\ &\leq \inf_{K \in \mathbb{N}} E[-X 1_{\{X > -K\}}] + \text{AVaR}_\alpha(X 1_{\{X \leq -K\}}) \\ &\leq E[-X] + \lim_{K \rightarrow \infty} \text{AVaR}_\alpha(X 1_{\{X \leq -K\}}) = E[-X] \end{aligned}$$

because  $\text{AVaR}_\alpha$  is continuous and  $X 1_{\{X \leq -K\}} \rightarrow 0$  in  $L^1$  for  $K \rightarrow \infty$ . This proves (6.10).

Now, choose any  $X \in L^1$  being unbounded from below. Suppose there is an optimal allocation  $(X_1, X_2)$  of  $X$ . Then  $X_1$  must be bounded and  $X_2$  unbounded from below, respectively. In view of lemma 6.3, we thus have  $\text{AVaR}_\alpha(X_2) > E[-X_2]$ , and hence

$$\begin{aligned} \rho_1 \square \rho_2(X) &= \rho_1(X_1) + \rho_2(X_2) \\ &= E[-X_1] + \delta(X_1^- | L^\infty) + \text{AVaR}_\alpha(X_2) \\ &> E[-X_1] + E[-X_2] = E[-X], \end{aligned}$$

which contradicts (6.10). Hence, there exists no optimal allocation of  $X$ .  $\parallel$

As announced in remark 2.7, optimal allocations are not unique in general. A trivial example is

$$(-E) \square (-E) = -E.$$

In this case, all allocations of any  $X \in L^1$  are optimal allocations of  $X$ . In particular, all comonotone allocations of  $X$  are optimal. Thus, even amongst the comonotone allocations, we cannot expect uniqueness. The following example further illustrates this fact:

**Example 6.6.** Let  $0 \leq \beta \leq \gamma < 1$ . In view of (6.7), there exists  $\mathbb{Q} \in \text{dom AVaR}_\gamma^*$  and  $A \in \mathcal{F}$  such that  $\mathbb{Q}(A) = 0 < \mathbb{P}(A)$ . Properties (6.9) and (6.8) imply  $\mathbb{Q} \in \text{dom AVaR}_\beta^*$ , and thus

$$\text{AVaR}_\beta \square \text{AVaR}_\gamma(1_A) = \text{AVaR}_\gamma(1_A) = 0 = \text{AVaR}_\beta(1_A).$$

Hence, both  $(1_A, 0)$  and  $(0, 1_A)$  are optimal allocations of  $1_A$ .  $\parallel$

On the other hand, there are situations in which there is uniqueness up to rebalancing the cash, such as the following:

**Example 6.7.** Let  $\beta \in (0, 1)$ . We know that  $\text{AVaR}_\beta \square -E = -E$  (example 6.4). Suppose  $(Y, X - Y)$  is an optimal allocation of  $X$ . This implies  $\text{AVaR}_\beta(Y) + E[-(X - Y)] = E[-X]$ , that is,  $\text{AVaR}_\beta(Y) = E[-Y]$ . In view of lemma 6.3 we conclude that  $Y$  must be constant, i.e. a cash position.  $\parallel$

## A Some Facts from Convex Analysis

For the convenience of the reader we collect here some standard definitions and results in convex analysis. For more background we refer to Rockafellar [21] and Ekeland and Témam [10].

Let  $E$  denote a Hausdorff locally convex topological vector space with topological dual  $E^*$ . A function  $f : E \rightarrow [-\infty, +\infty]$  is *convex* if

$$f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y) \quad \forall X, Y \in E, \quad \forall \lambda \in [0, 1],$$

whenever the right-hand side is defined. We write  $\text{dom } f = \{f < \infty\}$  for the (*effective*) *domain* of  $f$ . We call  $f$  *proper* if  $f > -\infty$  and  $\text{dom } f \neq \emptyset$ .

$f$  is said to be *lower semi-continuous* (l.s.c.) if the level sets  $\{X \in E \mid f(X) \leq k\}$  are closed for all  $k \in \mathbb{R}$ , or equivalently, if for any net  $(X_\alpha)_{\alpha \in D} \subset E$  converging to some  $X \in E$  we have that  $f(X) \leq \liminf_\alpha f(X_\alpha)$ . This property is also equivalent to  $\text{epi } f = \{(X, a) \in E \times \mathbb{R} \mid f(X) \leq a\}$  being a closed set in  $E \times \mathbb{R}$  equipped with the product topology (see e.g. [10] proposition 2.3).

A convex set  $\mathcal{C} \subset E$  is closed if and only if it is  $\sigma(E, E^*)$ -closed. As a consequence, a convex function  $f$  is l.s.c. if and only if  $f$  is l.s.c. with respect to  $\sigma(E, E^*)$ .

The *closure* of  $f$  is denoted by  $\text{cl}(f)$  and defined as  $\text{cl}(f) \equiv -\infty$ , if  $f(X) = -\infty$  for some  $X$ , and as greatest convex l.s.c. function majorised by  $f$ , else.

The *conjugate function* of a function  $f : E \rightarrow [-\infty, +\infty]$ ,

$$f^*(\mu) = \sup_{X \in E} (\langle \mu, X \rangle - f(X)),$$

is a l.s.c. convex function on  $E^*$ . Moreover,  $(\text{cl}(f))^* = f^*$ , and the following convex duality relation holds (proposition 4.1 in [10])

$$f^{**} = \text{cl}(f). \tag{A.11}$$

The *indicator function* of a set  $\mathcal{C} \subset E$  is defined as

$$\delta(X \mid \mathcal{C}) := \begin{cases} 0, & X \in \mathcal{C} \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

$\delta(\cdot \mid \mathcal{C})$  is convex and l.s.c. if and only if  $\mathcal{C}$  is convex and closed. Its conjugate is the *support function* of  $\mathcal{C}$ ,

$$\delta^*(\mu \mid \mathcal{C}) = \sup_{X \in \mathcal{C}} \langle \mu, X \rangle.$$

Notice that  $E$  and  $E^*$  can be interchanged in the definition of  $\delta$  and  $\delta^*$ .

## B Law-invariant Convex Functions on $L^p$

The proof of theorem 2.5 draws heavily on the following properties, which are proved in the appendix of [15]. Let  $F : L^p \rightarrow (-\infty, \infty]$  be a proper l.s.c. convex function, then the following conditions are equivalent:

$$F \text{ is law-invariant} \Leftrightarrow F^* \text{ is law-invariant} \tag{B.12}$$

$$\Leftrightarrow X \succeq_c Y \text{ implies } F(X) \leq F(Y). \tag{B.13}$$

Consequently, by Jensen's inequality, if  $F$  is law-invariant, then

$$F(E[X \mid \mathcal{G}]) \leq F(X) \text{ for all sub-}\sigma\text{-algebras } \mathcal{G} \subset \mathcal{F}. \tag{B.14}$$

Now let  $\rho : L^p \rightarrow (-\infty, \infty]$  be a l.s.c. law-invariant convex cash-invariant function. Then cash-invariance and (B.14) imply that

$$\rho(X) \geq -E[X] + \rho(0). \tag{B.15}$$

Clearly,

$$\rho^*(-1) = \sup_{X \in L^p} E[-X] - \rho(X) \geq -\rho(0).$$

However, if  $\rho^*(-1) > -\rho(0)$ , there must be some  $Y \in L^p$  such that  $E[-Y] - \rho(Y) > -\rho(0)$ . But this contradicts (B.15). We thus have shown

$$\rho^*(-1) = -\rho(0). \tag{B.16}$$

## C An Arzela-Ascoli Type Argument

The following lemma is needed for the proofs in sections 4 and 5.

**Lemma C.1.** *Let  $f_n : \mathbb{R} \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , be a sequence of increasing 1-Lipschitz-continuous functions such that  $f_n(0) \in [-K, K]$  for all  $n \in \mathbb{N}$  where  $K \geq 0$  is a constant. Then there is a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  and an increasing 1-Lipschitz-continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$  for all  $x \in \mathbb{R}$ .*

*Proof.* The Lipschitz-continuity guarantees that  $f_n(x) \in [-K, K + x]$  if  $x \geq 0$  and  $f_n(x) \in [-K + x, K]$  if  $x \leq 0$ . Hence, by a procedure well-known from the standard proof of the Arzela-Ascoli theorem, we are able to extract a subsequence  $(f_{n_k})_{k \in \mathbb{N}}$  of  $(f_n)_{n \in \mathbb{N}}$  such that  $\lim_{k \rightarrow \infty} f_{n_k}(q)$  exists for all  $q \in \mathbb{Q}$ . In fact, we can easily show that the sequences  $(f_{n_k}(x))_{k \in \mathbb{N}}$  must converge for all  $x \in \mathbb{R}$ . To this end, let  $\epsilon > 0$  be arbitrary and choose  $q \in \mathbb{Q}$  and  $N_0 \in \mathbb{N}$  such that  $|q - x| < \epsilon/3$  and  $|f_{n_k}(q) - f_{n_l}(q)| < \epsilon/3$  for all  $k, l \geq N_0$ . Then for all  $k, l \geq N_0$ :

$$\begin{aligned} |f_{n_k}(x) - f_{n_l}(x)| &\leq |f_{n_k}(x) - f_{n_k}(q)| + |f_{n_k}(q) - f_{n_l}(q)| + \\ &\quad + |f_{n_l}(q) - f_{n_l}(x)| \\ &\leq 2|x - q| + |f_{n_k}(q) - f_{n_l}(q)| < \epsilon, \end{aligned}$$

in which we did apply the Lipschitz-continuity twice. Now it is easily verified that  $f(x) := \lim_{k \rightarrow \infty} f_{n_k}(x)$ ,  $x \in \mathbb{R}$ , is a 1-Lipschitz-continuous increasing function.  $\square$

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