

Equilibrium Prices for Monetary Utility Functions*

Damir Filipović and Michael Kupper

Mathematics Institute

University of Munich

Theresienstrasse 39, D-80333 Munich, Germany

filipo@math.lmu.de, kupper@math.ethz.ch

20 November 2006

Abstract

This paper provides sufficient and necessary conditions for the existence of equilibrium pricing rules for monetary utility functions under convex consumption constraints. These utility functions are characterized by the assumption of a fully fungible numeraire asset (“cash”). Each agent’s utility is nominally shifted by exactly the amount of cash added to his endowment. We find the individual maximum utility that each agent is eligible for in an equilibrium and provide a game theoretic point of view for the fair allocation of the aggregate utility.

Key words: existence of equilibrium prices, monetary utility functions, Pareto optimal allocation, convex consumption constraints

1 Introduction

Monetary utility functions are characterized by the assumption of a fully fungible numeraire asset (“cash”) and the property that an agent’s utility is nominally shifted by exactly the amount of cash added to his endowment. This “cash invariance” introduces the possibility of “rebalancing of the cash” without restrictions at any time.

Jouini, Schachermayer and Touzi [20] provide an existence result for Pareto optimal allocations in the case of law-invariant monetary utility functions. We extend their framework and consider an infinite dimensional economy, where the agents are described by convex cash invariant unbounded below consumption sets and monetary utility functions. It is well known that equilibria do not exist in general in such a framework (see [20] and [12] for explicit counter examples. We also refer to [9] for an overview of the economics

*We thank Patrick Cheridito, Freddy Delbaen, Michael Florig, Walter Schachermayer, and Gregor Svindland for helpful comments. Filipović gratefully acknowledges the hospitality of the School of Mathematics and Statistics, UNSW, Sydney. Kupper gratefully acknowledges financial support by the Swiss National Science Foundation and Credit Suisse.

literature on this topic). On the other hand, existence results such as in [4, 7, 9, 26] do not apply for monetary utility functions, as we will make clear below in Section 3.

This leads us to develop another concept, that of an equilibrium pricing rule. Such pricing rules support uniformly asymptotic optimal allocations. We provide sufficient and necessary conditions for the existence of equilibrium pricing rules. This applies in particular if a financial market is assumed (Example 3.9). Moreover, in our setup, we show that the existence of an equilibrium is equivalent to the existence of a Pareto optimal allocation. The equilibrium pricing rules coincide with the super-gradients of the representative agent's utility function (Remark 4.5). We make essential use of the fact that this utility function does not depend on the individual agents' initial endowments (it is just the convolution of the individual utility functions) and that the set of super-gradients is not empty if it is finite valued. For each pricing rule we then can calculate the maximum utility that each agent is qualified for given his initial endowment. Moreover, we provide a game theoretic point of view for the fair allocation of the aggregate utility.

Assuming the space L^∞ of essentially bounded payoffs, the pricing rules are only finitely additive in general. We therefore provide sufficient conditions, for the unconstrained case, under which the equilibrium pricing rules are given as expectation operators and thus are σ -additive.

Monetary risk measures have recently attracted much attention in the mathematical finance community, see e.g. [2, 3, 17]. Monetary utility functions are, up to the sign, identical to convex risk measures. Heath and Ku [18] characterize Pareto optimality for convex risk measures in a simpler framework without constraints, similar to the setup in [3]. That characterization is extended in Burgert and Rüschemdorf [5]. In particular, they introduce trading constraints described by a linear subspace. Equilibria for positively homogeneous convex (this is, coherent) risk measures in connection with financial markets have also been recently considered in [6]. Our paper is more general and encompasses [18, 5, 6] in that it considers general concave monetary utility functions including convex trading constraints. Moreover, we provide explicit conditions for the existence of equilibrium pricing rules.

Monetary utility functions can be seen as particular type of niveloid in the spirit of [13] and [21]. Within the traditional class of expected utility functions, only the affine and entropic utility functions are monetary. In fact, the latter is defined as the scaled logarithm of the exponential utility (see [17]), which induces the same preference order as $-$ and is thus equivalent to $-$ the exponential utility function. However, extending the class of expected utility functions to multi-prior utility functions enlarges the intersection with the class of monetary utility functions. Eventually, all monetary utility functions belong to a general class of utility functions recently introduced by Maccheroni, Marinacci and Rustichini in [21]. A typical example of preferences induced by a monetary utility function is the monetary version of the mean-variance preferences of Markovitz and Tobin (see for instance [22]). A detailed study concerning equilibria and Pareto optimality for monetary utility functions under constraints, such as provided by this paper, is therefore needed. Notice that by switching the sign, which turns utility functions into risk measures, our results can also be used to do derive fair values for optimal risk exchanges subject to

constraints (see for instance [15]).

The outline of the paper is as follows. In Section 2 we introduce the notion of a monetary utility function restricted to a convex cash invariant subset of L^∞ . From convex duality theory there results a representation of any monetary utility function in terms of its conjugate function. It is key that the inf in the representation is always attained (Proposition 2.5). The section concludes with the definition and basic properties of super-gradients of monetary utility functions.

Section 3 contains our main results. We introduce the economy and define equilibrium pricing rules. As illustrative example we consider the case with a financial market. We then provide sufficient and necessary conditions for the existence of equilibrium pricing rules. Moreover, we show that the existence of an equilibrium is equivalent to the existence of a Pareto optimal allocation in our framework. Finally, we provide a game theoretic point of view on the optimality of an equilibrium.

Section 4 contains the key results on the constrained convolution of monetary utility functions, which are at the core for the proofs of the main theorems. It also briefly discusses the representative agent view.

In Section 5 we characterize Pareto optimality and resolve some issues regarding the market clearing.

Section 6 provides, for the unconstrained case, sufficient conditions for the utility functions such that the equilibrium pricing rules are σ -additive. It is also illustrated by an example how these conditions may fail under constraints.

For the sake of readability, some proofs are postponed to the appendix.

2 Monetary utility functions on subsets of L^∞

Throughout this paper, we fix a probability space (Ω, \mathcal{F}, P) . All equalities and inequalities between random variables are always understood in the P -almost sure sense.

L^∞ and L^1 denote the Banach spaces of all essentially bounded and integrable random variables, respectively, where random variables which are P -almost surely equal are identified. $(L^\infty)^*$ denotes the dual space of L^∞ . It is well known, that $(L^\infty)^*$ can be identified with the space of all bounded finitely additive measures on (Ω, \mathcal{F}) which are absolutely continuous with respect to P . We shall write

$$(L^\infty)_+^* := \{\mu \in (L^\infty)^* \mid \langle \mu, \xi \rangle \geq 0 \forall \xi \geq 0\} \quad \text{and} \quad \mathcal{P} := \{\mu \in (L^\infty)_+^* \mid \langle \mu, 1 \rangle = 1\}. \quad (2.1)$$

\mathcal{P} is the convex set of *pricing rules*.

A function $f : L^\infty \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ is *proper* if $f < \infty$ and its *domain*

$$\text{dom}(f) := \{\xi \in L^\infty \mid f(\xi) > -\infty\}$$

is non-empty (since in this paper we consider concave functions, the signs in this definition are different than for convex functions as in [14, 23]). For a set $M \subseteq L^\infty$, we define the *restriction of f to M* as

$$f^M(\xi) := \begin{cases} f(\xi), & \xi \in M \\ -\infty, & \text{else.} \end{cases}$$

The *conjugate function* of f is defined by

$$f^*(\mu) := \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - f(\xi)), \quad \mu \in (L^\infty)^*,$$

and the set of *super-gradients* is denoted by

$$\partial f(\xi) = \{\mu \in (L^\infty)^* \mid f(\eta) \leq f(\xi) + \langle \mu, \eta - \xi \rangle \text{ for every } \eta \in L^\infty\}, \quad \xi \in L^\infty.$$

The following characterization is fundamental (see e.g. [14, Proposition 5.1])

$$\mu \in \partial f(\xi) \quad \Leftrightarrow \quad f(\xi) + f^*(\mu) = \langle \mu, \xi \rangle. \quad (2.2)$$

If f is proper, concave and $\sigma(L^\infty, (L^\infty)^*)$ -upper semi-continuous then $f^* : (L^\infty)^* \rightarrow \overline{\mathbb{R}}$ is proper, concave and $\sigma((L^\infty)^*, L^\infty)$ -upper semi-continuous and

$$f^{**}(\xi) := \inf_{\mu \in (L^\infty)^*} (\langle \mu, \xi \rangle - f^*(\mu)) = f(\xi),$$

see e.g. [14, Proposition 4.1, Chapter I].

Definition 2.1 A proper function $U : L^\infty \rightarrow \overline{\mathbb{R}}$ is called *monetary utility function* if it is

- (i) *monotone*: $U(X) \geq U(Y)$ if $X \geq Y$,
- (ii) *concave*: $U(\lambda X + (1 - \lambda)Y) \geq \lambda U(X) + (1 - \lambda)U(Y)$ for all $\lambda \in [0, 1]$,
- (iii) *cash invariant*: $U(X + m) = U(X) + m$ for all $m \in \mathbb{R}$.

Remark 2.2 For a monetary utility function U we thus have $\partial U(\xi) \subset \mathcal{P}$ for all $\xi \in L^\infty$. Indeed, for every $\mu \in \partial U(\xi)$ we have $\langle \mu, \eta \rangle \geq U(\xi + \eta) - U(\xi)$ for all $\eta \in L^\infty$. For $\eta \equiv c \in \mathbb{R}$ we obtain $c\langle \mu, 1 \rangle \geq c$, which implies $\langle \mu, 1 \rangle = 1$. Moreover, $0 \leq U(\xi + \eta) - U(\xi) \leq \langle \mu, \eta \rangle$ for all $\eta \geq 0$ implies $\mu \in (L^\infty)_+^*$.

It also follows from the monotonicity and cash invariance that U is \mathbb{R} -valued and Lipschitz continuous with respect to the L^∞ -norm, and hence $\sigma(L^\infty, (L^\infty)^*)$ -upper semi-continuous.

Remark 2.3 U is a monetary utility function if and only if $\rho = -U$ is a convex risk measure on L^∞ . For a detailed discussion of convex risk measures we refer to [17] and references therein. Convex risk measures are a generalization of coherent risk measures, which were introduced to the mathematical finance literature in [2, 10].

Definition 2.4 A non-empty set $M \subseteq L^\infty$ is called *cash invariant* if $X \in M$ implies $X + m \in M$ for all $m \in \mathbb{R}$.

The following proposition summarizes the crucial properties of the restriction U^M of U to a cash-invariant set M . The proof is postponed to Section A.

Proposition 2.5 *Let U be a monetary utility function and $M \subseteq L^\infty$ be a convex closed cash invariant set. Then U^M is proper, concave, cash invariant, $\sigma(L^\infty, (L^\infty)^*)$ -upper semi-continuous, and U^M can be represented by*

$$U^M(\xi) = \min_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^{M*}(\mu)) \quad \forall \xi \in M. \quad (2.3)$$

Moreover,

$$\sup_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} U^M(\eta) = \langle \mu, \xi \rangle - U^{M*}(\mu) \quad \forall \mu \in \mathcal{P}, \quad \forall \xi \in M, \quad (2.4)$$

and the set of super-gradients satisfies

$$\partial U^M(\xi) \cap \mathcal{P} \neq \emptyset \quad \forall \xi \in M, \quad (2.5)$$

$$\partial U^M(\xi) = \emptyset \quad \forall \xi \notin M. \quad (2.6)$$

Notice that U^M is not monotone in general. Monotonicity of risk measures is discussed in some detail in [16].

Property (2.4) distinguishes monetary utility functions and is key for what follows below. Indeed, it is easy to find non-monetary examples (e.g. $\Omega = \{\omega\}$, $L^\infty \equiv \mathbb{R}$, $\mathcal{P} = \{1\}$ and $U^M(\xi) = \xi/2$) where (2.4) does not hold.

3 Equilibria and Pareto optimal allocations under constraints

We consider a pure exchange economy with $n \geq 2$ agents. Agent i is described by a convex closed cash invariant consumption set $M_i \subseteq L^\infty$ and an initial endowment $X_i \in M_i$. The preferences of agent i are represented by a monetary utility function U_i . We write

$$X := X_1 + \dots + X_n$$

for the aggregate endowment (market portfolio), and define $\mathbb{M} := M_1 \times \dots \times M_n \subseteq (L^\infty)^n$.

Definition 3.1 *An allocation (ξ_1, \dots, ξ_n) is called attainable if $(\xi_1, \dots, \xi_n) \in \mathbb{M}$ and the clearing condition $\sum_{i=1}^n \xi_i \leq X$ holds.*

Attainable allocations are the possible sharings of the aggregate endowment X among the n agents. Notice that $P[\xi_1 + \dots + \xi_n < X] > 0$ is allowed, which amounts to say that markets clear up to a non-negative residual. In economic terms this is usually referred to as “free disposal”. We provide an interpretation in the context of optimal capital and risk sharing in [15]. See also Remark 5.2 below for a more formal interpretation.

Definition 3.2 *An attainable allocation (ξ_1, \dots, ξ_n) together with a pricing rule $\mu \in \mathcal{P}$ is called an equilibrium if*

$$U_i(\xi_i) = \sup_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle, \eta \in M_i} U_i(\eta) \quad \text{and} \quad \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad i = 1, \dots, n. \quad (3.7)$$

Hence each agent i optimizes his utility subject to his consumption ($\xi_i \in M_i$) and budget ($\langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle$) constraints.

Existence theorems for equilibria have been given in [4] for bounded below consumption sets on L^∞ , and in [26] for unbounded below consumption sets on a finite dimensional model space. On the other hand, it is well known that equilibria do not exist in general in the present infinite dimensional setup where consumption sets M_i are unbounded below (see [20] and [12] for explicit counter examples, and [9] for an overview of the economics literature on this topic). Existence results such as in Dana et al. [7, 9] do not apply in our framework. Indeed, monetary utility functions are not of mean variance type as in [7]. Moreover, one can show that the set of the so-called fair utility weight vectors in [9] is always empty for monetary utility functions (in the notation of [9] it follows that “ $U(0)$ ” is of the form $\{v \in \mathbb{R}^m \mid \langle v, 1 \rangle \leq c\}$ for some $c \geq 0$; therefore the polar cone “ D ” is generated by $(1, \dots, 1)$ and thus has empty interior).

We now introduce the concept of a pricing rule which supports a uniformly asymptotic equilibrium.

Definition 3.3 *We call $\mu \in \mathcal{P}$ an equilibrium pricing rule if, for every $\varepsilon > 0$, there exists an ε -equilibrium, that is, an attainable allocation (ξ_1, \dots, ξ_n) such that*

$$\langle \mu, X_i \rangle - \varepsilon \leq \langle \mu, \xi_i \rangle \leq \langle \mu, X_i \rangle \quad \text{and} \quad (3.8)$$

$$\sup_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle, \eta \in M_i} U_i(\eta) \leq U_i(\xi_i) + \varepsilon \quad \forall i = 1, \dots, n. \quad (3.9)$$

Note that ε -optimality (3.9) is obtained uniformly across all agents, while the budget constraints (3.8) apply. Moreover, $(\xi_1, \dots, \xi_n, \mu)$ is an equilibrium only if μ is an equilibrium pricing rule. Hence, in order that an equilibrium allocation ever be found, the agents have to trade with respect to an equilibrium pricing rule. It is therefore vital to have a characterization and existence result for the set of equilibrium pricing rules.

Definition 3.4 *The \mathbb{M} -constrained convolution of U_1, \dots, U_n is defined by*

$$\square_i^{\mathbb{M}} U_i(\xi) := \sup_{\sum_i \xi_i \leq \xi, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i), \quad \xi \in L^\infty. \quad (3.10)$$

The unconstrained convolution of U_1, \dots, U_n (that is, (3.10) for $M_i = L^\infty$) is simply denoted by $\square_i U_i$.

Here is our main existence result.

Theorem 3.5 *The set of equilibrium pricing rules equals $\partial \square_i^{\mathbb{M}} U_i(X) \subset \mathcal{P}$. Moreover, there exists an equilibrium pricing rule $\mu \in \partial \square_i^{\mathbb{M}} U_i(X)$ if and only if one of the following equivalent conditions holds:*

$$\bigcap_{i=1}^n \text{dom} \left(U_i^{M_i^*} \right) \neq \emptyset, \quad \text{or} \quad (3.11)$$

$$\square_i^{\mathbb{M}} U_i(X) < \infty. \quad (3.12)$$

In this case, the individual maximum utility is given by

$$\sup_{\langle \mu, \eta \rangle \leq \langle \mu, X_i \rangle, \eta \in M_i} U_i(\eta) = \langle \mu, X_i \rangle - U_i^{M_i^*}(\mu). \quad (3.13)$$

Proof. This follows from (2.4), (2.5) and Lemmas 4.2–4.4 below. \square

Note that the right hand side of (3.13) gives an *a priori* value for the individual maximum utility agent i is eligible for in an equilibrium.

The following corollary gives sufficient conditions for the existence of an equilibrium pricing rule in terms of the unconstrained utility functions U_i .

Corollary 3.6 *Any of the following two equivalent conditions implies existence of an equilibrium pricing rule:*

$$\bigcap_{i=1}^n \text{dom}(U_i^*) \neq \emptyset, \quad \text{or} \quad (3.14)$$

$$\square_i U_i(X) < \infty. \quad (3.15)$$

Proof. Equivalence of (3.14) and (3.15) follows from Theorem 3.5 for $M_i = L^\infty$.

Moreover, $U_i^* \leq U_i^{M_i^*} < \infty$ and therefore $\text{dom}(U_i^*) \subseteq \text{dom}(U_i^{M_i^*})$. Hence the corollary follows from (3.11). \square

Corollary 3.7 *Suppose (Ω, \mathcal{F}, P) is atomless and U_i is law-invariant (that is, $U_i(\xi) = U_i(\eta)$ for all $\xi \stackrel{(d)}{=} \eta$) for all $i = 1, \dots, n$. Then there exists an equilibrium pricing rule.*

Proof. It is shown in [20] that under the stated assumptions, $1 \in \text{dom}(U_i^*)$ for all i . Hence the claim follows from Corollary 3.6. \square

Dana and Le Van [9] provide sufficient conditions for the existence of “quasi-equilibria”, a stronger concept than ε -equilibria, which implies the existence of equilibrium pricing rules in particular. However, by the same arguments as those after Example 3.9, these conditions do not apply for our monetary setup. In fact, our monetary setup can be considered as limit case (where the polar cone “D” in [9] is generated by the vector $(1, \dots, 1)$), and therefore as a completion, of [9].

Here is a simple example where an equilibrium pricing rule does not exist.

Example 3.8 Let $n = 2$ and fix two random variables $D_1 \neq D_2$ with $D_i \geq 0$ and $E[D_i] = 1$. Define the monetary utility functions $U_i(\xi) := E[D_i \xi]$ and let $M_i = L^\infty$. Then $\text{dom}(U_i^*) = \{D_i\}$, so that (3.11) is not satisfied.

The above setup is very general. For the sake of illustration we should have the following example with a financial market in mind.

Example 3.9 Assume there exists a financial market consisting of $m + 1$ securities with discounted payoffs $S_0, S_1, \dots, S_m \in L^\infty$, where $S_0 \equiv 1$ is the numeraire asset. We let

$$D := \left\{ \sum_{j=0}^m \beta_j S_j \mid \beta_j \in \mathbb{R} \right\}$$

denote the space of attainable payoffs by trading. The consumption set of agent i is then defined as

$$M_i = X_i + D, \quad (3.16)$$

that is, any attainable allocation is now of the form $\xi_i = X_i + \sum_{j=0}^m \beta_{i,j} S_j$.

Now suppose that (3.14) holds. Then Corollary 3.6 asserts the existence of equilibrium prices

$$s_j := \langle \mu, S_j \rangle \quad j = 0, \dots, m$$

of the securities in the financial market, for some equilibrium pricing rule $\mu \in \partial \square_i^M U_i(X)$. The budget constraint for agent i in (3.7) boils down to

$$\sum_{j=0}^m \beta_{i,j} s_j \leq 0, \quad i = 1, \dots, n,$$

and by individual utility maximization (3.7), the economy will eventually come to an equilibrium.

This framework can be extended to include individual convex trading (such as short-selling) constraints by replacing D in (3.16) by

$$D_i := \left\{ \sum_{j=0}^m \beta_j S_j \mid \beta_0 \in \mathbb{R}, (\beta_1, \dots, \beta_m) \in W_i \right\},$$

for some closed convex set $W_i \subseteq \mathbb{R}^m$ (note that X_i may already contain shares of S_0, \dots, S_m).

Alternatively, we may define the above consumption sets as

$$M_i = \{ \xi \in X_i + D_i \mid \mathbb{E}[(\xi - \mathbb{E}[\xi])_-] \leq \kappa_i \}$$

for some risk bound $\kappa_i \geq 0$ such that $\mathbb{E}[(X_i - \mathbb{E}[X_i])_-] \leq \kappa_i$. The interpretation is clear: every agent faces individual trading constraints in terms of size (D_i) and riskiness (κ_i) of its portfolio.

This example can be further extended by replacing $\mathbb{E}[(\xi - \mathbb{E}[\xi])_-]$ by a so called generalized deviation measure $\mathcal{D}_i : L^\infty \rightarrow \mathbb{R}_+$, recently introduced in [24] (see also [16] for more examples), such that $\xi \mapsto \rho_i(\xi) := \mathbb{E}[-\xi] + \mathcal{D}_i(\xi)$ becomes a coherent risk measure. In any case, our results assert the existence of equilibrium prices.

3.1 Pareto optimality

Strongly linked to equilibrium is another optimality concept:

Definition 3.10 *An attainable allocation (ξ_1, \dots, ξ_n) is called Pareto optimal if for all attainable (η_1, \dots, η_n) we have*

$$U_i(\eta_i) \geq U_i(\xi_i) \quad \forall i = 1, \dots, n \quad \text{only if} \quad U_i(\eta_i) = U_i(\xi_i) \quad \forall i = 1, \dots, n.$$

It is a classical result from the economic theory (Welfare Theorem) that an equilibrium is Pareto optimal. Conversely, a Pareto optimal allocation is an equilibrium up to transfer payments (see e.g. [8]). Because of the cash-invariance of U_i and M_i we have in fact equivalence, as stated by the following theorem, the proof of which is given in Section B.

Theorem 3.11 *$(\xi_1, \dots, \xi_n, \mu)$ is an equilibrium if and only if*

- (i) (ξ_1, \dots, ξ_n) is Pareto optimal and
- (ii) $\mu \in \partial \square_i^{\mathbb{M}} U_i(X)$ and
- (iii) $\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$ for all $i = 1, \dots, n$.

Conversely, for every Pareto optimal allocation (ξ_1, \dots, ξ_n) and $\mu \in \partial \square_i^{\mathbb{M}} U_i(X)$ we have that

$$(\xi_1 + \langle \mu, X_1 - \xi_1 \rangle, \dots, \xi_n + \langle \mu, X_n - \xi_n \rangle, \mu) \quad \text{is an equilibrium.} \quad (3.17)$$

Remark 3.12 It is shown in Theorem 5.1 below, see (5.29), that the existence of a Pareto optimal allocation implies the existence of an equilibrium pricing rule $\mu \in \partial \square_i^{\mathbb{M}} U_i(X)$ and thus of an equilibrium.

On the other hand, there are situations where an equilibrium pricing rule exists, but an equilibrium allocation is not attained, see [20, 12] for examples.

For the case $M_i = L^\infty$, it is shown in [20] that Pareto optimal allocations exist if U_1, \dots, U_n are law-invariant; that is, $U_i(\xi) = U_i(\eta)$ for all $\xi \stackrel{(d)}{=} \eta$.

3.2 Game theoretic view

In this section, we show that no subset (coalition) of agents can overturn the aggregate market utility of the entire economy. Indeed, we obtain a ‘‘fair allocation’’ of the aggregate market utility across the agents.

An allocation $(k_1, \dots, k_n) \in \mathbb{R}^n$ is said to be in the *core of the game* with *characteristic function*

$$\{1, \dots, n\} \supseteq S \mapsto \square_{i \in S}^{\mathbb{M}} U_i \left(\sum_{i \in S} X_i \right) := \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i, \eta_i \in M_i} \sum_{i \in S} U_i(\eta_i) \quad (3.18)$$

if

$$\sum_{i=1}^n k_i = \square_i^{\mathbb{M}} U_i(X) \quad \text{and} \quad (3.19)$$

$$\sum_{i \in S} k_i \geq \square_{i \in S}^{\mathbb{M}} U_i \left(\sum_{i \in S} X_i \right) \quad \text{for all } S \subseteq \{1, \dots, n\}. \quad (3.20)$$

Property (3.19) is simply the Pareto optimality, as we shall see in Theorem 5.1, Property (ii), below. We now obtain a stronger version:

Corollary 3.13 *Let $(\xi_1, \dots, \xi_n, \mu)$ be an equilibrium. Then*

$$(U_1(\xi_1), \dots, U_n(\xi_n)) = (\langle \mu, X_1 \rangle - U_1^{M_1^*}(\mu), \dots, \langle \mu, X_n \rangle - U_n^{M_n^*}(\mu)) \quad (3.21)$$

lies in the core of the game (3.18).

Proof. Property (3.19) and (3.21) follow from Theorem 3.11 and (2.4). For (3.20) we simply calculate

$$\begin{aligned} \sum_{i \in S} (\langle \mu, X_i \rangle - U_i^{M_i^*}(\mu)) &= \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i, \eta_i \in M_i} \sum_{i \in S} (\langle \mu, \eta_i \rangle - U_i^{M_i^*}(\mu)) \\ &\geq \sup_{\sum_{i \in S} \eta_i \leq \sum_{i \in S} X_i, \eta_i \in M_i} \sum_{i \in S} U_i(\eta_i) = \square_{i \in S}^{\mathbb{M}} U_i \left(\sum_{i \in S} X_i \right). \end{aligned}$$

□

Coherent utility function based allocation methods are further discussed in [11].

4 On the \mathbb{M} -constrained convolution

The unconstrained convolution, $\square_i U_i$, of U_1, \dots, U_n (see Definition 3.4) has been studied in e.g. [3]. By monotonicity of U_i one easily sees that

$$\square_i U_i(\xi) = \sup_{\sum_i \xi_i \leq \xi} \sum_{i=1}^n U_i(\xi_i) = \sup_{\sum_i \xi_i = \xi} \sum_{i=1}^n U_i(\xi_i). \quad (4.22)$$

However, Example 4.1 below shows that

$$\square_i^{\mathbb{M}} U_i(\xi) > \sup_{\sum_i \xi_i = \xi, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i) \quad (4.23)$$

is possible, even if $\xi \in \sum_i M_i$. Hence a naive extension of (4.22) to the constrained case is not possible. On the other hand, it follows by inspection that we have equality in (4.23) if all $M_i \equiv M$ coincide and are linear, which is the assumption made in [5].

Example 4.1 Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $L^\infty \cong \mathbb{R}^3$, and $n = 2$. Consider the linear independent random variables $X_0 = (1, 1, 1)$, $X_1 = (1, 0, 0)$, $X_2 = (-1, 0, 1)$, and the convex closed cash invariant sets

$$M_1 = \text{span}\{X_0, X_1\}, \quad M_2 = \text{span}\{X_0, X_2\}.$$

The aggregate endowment is $X = (0, 0, 1) = X_1 + X_2$. Note that this decomposition is unique. Let U_i be the worst case utility function, i.e. $U_i(\xi) = \min\{\xi(\omega_1), \xi(\omega_2), \xi(\omega_3)\}$. We then calculate

$$\sup_{\xi_1 + \xi_2 = X, \xi_i \in M_i} (U_1(\xi_1) + U_2(\xi_2)) = U(X_1) + U(X_2) = 0 - 1 = -1.$$

On the other hand, since $0 \leq X$ we get

$$\sup_{\xi_1 + \xi_2 \leq X, \xi_i \in M_i} (U_1(\xi_1) + U_2(\xi_2)) \geq U_1(0) + U_2(0) = 0.$$

This proves (4.23).

The following key lemmas provide a characterization and the necessary and sufficient conditions under which the \mathbb{M} -constrained convolution is well-behaved. This is an extension of [20, Lemma 2.1].

Lemma 4.2 *The \mathbb{M} -constrained convolution $\square_i^{\mathbb{M}} U_i : L^\infty \rightarrow \overline{\mathbb{R}}$ is monotone, concave and cash-invariant, and its conjugate function is*

$$(\square_i^{\mathbb{M}} U_i)^*(\mu) = \begin{cases} \sum_{i=1}^n U_i^{M_i^*}(\mu), & \text{if } \mu \in (L^\infty)_+^* \\ -\infty, & \text{else.} \end{cases} \quad (4.24)$$

Proof. Monotonicity and concavity of $\square_i^{\mathbb{M}} U_i$ is obvious (e.g. the proof of [23, Theorem 5.4] carries over to our setup). Cash invariance follows from cash invariance of $U_i^{M_i}$.

Let $\mu \in (L^\infty)^*$, then we have

$$\begin{aligned} (\square_i^{\mathbb{M}} U_i)^*(\mu) &= \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \square_i^{\mathbb{M}} U_i(\xi) \right) = \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \sup_{\sum_{i=1}^n \xi_i \leq \xi, \xi_i \in M_i} \sum_{i=1}^n U_i(\xi_i) \right) \\ &\leq \inf_{\xi \in \sum_{i=1}^n M_i} \left(\inf_{\sum_{i=1}^n \xi_i = \xi} \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i(\xi_i)) \right) \\ &= \sum_{i=1}^n \inf_{\xi_i \in M_i} (\langle \mu, \xi_i \rangle - U_i(\xi_i)) = \sum_{i=1}^n U_i^{M_i^*}(\mu). \end{aligned}$$

On the other hand, for $\mu \in (L^\infty)_+^*$, we have

$$\begin{aligned} (\square_i^{\mathbb{M}}U_i)^*(\mu) &= \inf_{\xi \in L^\infty} \left(\langle \mu, \xi \rangle - \square_i^{\mathbb{M}}U_i(\xi) \right) \\ &\geq \inf_{\xi \in L^\infty} \left(\inf_{\sum_{i=1}^n \xi_i \leq \xi, \xi_i \in M_i} \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i(\xi_i)) \right) \\ &= \sum_{i=1}^n \inf_{\xi_i \in M_i} (\langle \mu, \xi_i \rangle - U_i(\xi_i)) = \sum_{i=1}^n U_i^{M_i^*}(\mu). \end{aligned}$$

If $\mu \notin (L^\infty)_+^*$, then there exists $\xi \in L_+^\infty$ with $\langle \mu, \xi \rangle < 0$. But $\square_i^{\mathbb{M}}U_i(n\xi) \geq 0$, for all $n \geq 1$, and hence

$$(\square_i^{\mathbb{M}}U_i)^*(\mu) \leq \langle \mu, n\xi \rangle - \square_i^{\mathbb{M}}U_i(n\xi) \leq n\langle \mu, \xi \rangle \quad \forall n \geq 1.$$

This proves (4.24). \square

Lemma 4.3 *The set of equilibrium pricing rules equals $\partial \square_i^{\mathbb{M}}U_i(X) \subset \mathcal{P}$.*

Proof. That $\partial \square_i^{\mathbb{M}}U_i(X) \subset \mathcal{P}$ follows from the monotonicity and cash invariance as in Remark 2.2.

Now let $\mu \in \mathcal{P}$ be an equilibrium pricing rule, and let $\varepsilon > 0$. By definition, there exists an attainable allocation (ξ_1, \dots, ξ_n) satisfying (3.8) and (3.9). In view of (2.4) and (4.24), we thus have

$$\begin{aligned} \square_i^{\mathbb{M}}U_i(X) &\geq \sum_{i=1}^n U_i^{M_i}(\xi_i) \geq \sum_{i=1}^n \left(\langle \mu, X_i \rangle - U_i^{M_i^*}(\mu) - \varepsilon \right) \\ &\geq \langle \mu, X \rangle - (\square_i^{\mathbb{M}}U_i)^*(\mu) - n\varepsilon \geq \square_i^{\mathbb{M}}U_i(X) - n\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\square_i^{\mathbb{M}}U_i(X) = \langle \mu, X \rangle - (\square_i^{\mathbb{M}}U_i)^*(\mu)$, and hence $\mu \in \partial \square_i^{\mathbb{M}}U_i(X)$, see (2.2).

Conversely, let $\mu \in \partial \square_i^{\mathbb{M}}U_i(X) \subset \mathcal{P}$ and $\varepsilon > 0$. Then, in view of (2.2), $\square_i^{\mathbb{M}}U_i(X) = \langle \mu, X \rangle - (\square_i^{\mathbb{M}}U_i)^*(\mu)$ and there exists an attainable allocation (ξ'_1, \dots, ξ'_n) such that

$$\sum_{i=1}^n U_i(\xi'_i) + \varepsilon \geq \square_i^{\mathbb{M}}U_i(X) = \langle \mu, X \rangle - (\square_i^{\mathbb{M}}U_i)^*(\mu) \geq \sum_{i=1}^n \left(\langle \mu, X_i \rangle - U_i^{M_i^*}(\mu) \right), \quad (4.25)$$

where we have used (4.24). By rebalancing the cash, see (5.28) below, we can find an attainable allocation $(\xi''_1, \dots, \xi''_n)$ which satisfies (4.25) instead of (ξ'_1, \dots, ξ'_n) and

$$\langle \mu, X_i \rangle - U_i^{M_i^*}(\mu) \leq U_i(\xi''_i) + \frac{\varepsilon}{n} \leq \langle \mu, \xi''_i \rangle - U_i^{M_i^*}(\mu) + \frac{\varepsilon}{n} \quad \forall i = 1, \dots, n. \quad (4.26)$$

Hence $\langle \mu, X_i \rangle \leq \langle \mu, \xi''_i \rangle + \frac{\varepsilon}{n}$. From $\sum_i X_i \geq \sum_i \xi''_i$ we conclude that

$$\langle \mu, X_i \rangle - \frac{\varepsilon}{n} \leq \langle \mu, \xi''_i \rangle \leq \langle \mu, X_i \rangle + \frac{(n-1)\varepsilon}{n} \quad \forall i = 1, \dots, n. \quad (4.27)$$

It is then clear from (4.26), (4.27) and (2.4) that $\xi_i = \xi''_i - \frac{(n-1)\varepsilon}{n}$ satisfy (3.8) and (3.9). Thus μ is an equilibrium pricing rule. \square

Lemma 4.4 $\square_i^{\mathbb{M}}U_i$ is proper – and thus a monetary utility function – if and only if (3.11) or (3.12) holds.

Proof. It follows from the monotonicity and cash-invariance (Lemma 4.2) that $\square_i^{\mathbb{M}}U_i$ is proper if and only if (3.12) holds. Equivalence of (3.11) and (3.12) follows from (4.24) and the fact that

$$-\infty < \text{ess inf}(\xi - X) + \sum_{i=1}^n U_i^{M_i}(X_i) \leq \square_i^{\mathbb{M}}U_i(\xi) \leq \langle \mu, \xi \rangle - (\square_i^{\mathbb{M}}U_i)^*(\mu), \quad \forall \xi \in L^\infty.$$

□

Remark 4.5 In view of the preceding lemmas, we can interpret $\square_i^{\mathbb{M}}U_i$ as utility function of the representative agent. Indeed, under the assumption of (3.11) or (3.12), it follows from (2.2), (2.4) and Lemma 4.3 that μ is an equilibrium pricing rule if and only if it makes the representative agent not wanting to trade away from its endowment X :

$$\square_i^{\mathbb{M}}U_i(X) = \sup_{\langle \mu, Y \rangle \leq \langle \mu, X \rangle} \square_i^{\mathbb{M}}U_i(Y),$$

where the supremum is taken over all $Y \in L^\infty$.

5 Characterization of Pareto optimality

We consider the setup of Section 3 and provide some preliminary results of independent interest on Pareto optimality, which will be used for the proof of Theorem 3.11. We also resolve the sub-clearing of markets by introducing a maximally risk averse dummy agent.

First, notice that Pareto optimal allocations are not unique in our setup. Indeed, for every attainable allocation (ξ_1, \dots, ξ_n) we can arbitrarily *rebalance the cash* without changing the aggregate utility $\sum_{i=1}^n U_i(\xi_i)$. That is, for every constant vector of cash transitions $(c_1, \dots, c_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n c_i = 0$ we have that $(\xi_1 + c_1, \dots, \xi_n + c_n)$ is attainable and by the cash invariance of U_i

$$\sum_{i=1}^n U_i(\xi_i + c_i) = \sum_{i=1}^n U_i(\xi_i). \quad (5.28)$$

The following characterization result for Pareto optimality is an extension of [20, 5, 18].

Theorem 5.1 Let (ξ_1, \dots, ξ_n) be an attainable allocation and set $\xi_0 := X - \sum_{i=1}^n \xi_i \geq 0$. Then the following properties are equivalent:

- (i) (ξ_1, \dots, ξ_n) is Pareto optimal,
- (ii) $\square_i^{\mathbb{M}}U_i(X) = \sum_{i=1}^n U_i(\xi_i)$,
- (iii) $\langle \mu, \xi_0 \rangle = 0$ and $U_i(\xi_i) = \langle \mu, \xi_i \rangle - U_i^{M_i^*}(\mu)$, $\forall i = 1, \dots, n$, for some $\mu \in \mathcal{P}$,

- (iv) $\langle \mu, \xi_0 \rangle = 0$ and $\mu \in \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$,
- (v) $U_0(\xi_0) = 0$ and $\mu \in \partial U_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$, where $U_0(\xi) := \text{ess inf } \xi$ is the worst case utility function.
- (vi) $\mu \in \partial V_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n)$ for some $\mu \in \mathcal{P}$, where

$$V_0(\xi) := \begin{cases} 0, & \xi \in L_+^\infty \\ -\infty, & \text{else} \end{cases}$$

is the concave indicator function of L_+^∞ .

Moreover, any of the above properties implies

$$\begin{aligned} \partial \square_i^M U_i(X) &= \partial U_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n) \\ &= \partial V_0(\xi_0) \cap \partial U_1^{M_1}(\xi_1) \cap \dots \cap \partial U_n^{M_n}(\xi_n) \neq \emptyset. \end{aligned} \quad (5.29)$$

Proof. (i) \Rightarrow (ii): Suppose that $\sum_{i=1}^n U_i(\xi_i) < \square_i^M U_i(X)$. Then there exists an attainable allocation (η_1, \dots, η_n) with $\sum_{i=1}^n U_i(\xi_i) < \sum_{i=1}^n U_i(\eta_i)$. By rebalancing the cash we can find an attainable allocation $(\eta'_1, \dots, \eta'_n)$ such that $U_1(\xi_1) < U_1(\eta'_1)$ and $U_i(\xi_i) \leq U_i(\eta'_i)$ for all $i = 2, \dots, n$. But then (ξ_1, \dots, ξ_n) is not Pareto optimal.

(ii) \Rightarrow (i) follows from the definition of Pareto optimality.

(iii) \Rightarrow (ii): follows from

$$\begin{aligned} \square_i^M U_i(X) &\geq \sum_{i=1}^n U_i(\xi_i) = \langle \mu, \xi_0 \rangle + \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i^{M_i^*}(\mu)) \\ &= \langle \mu, X \rangle - (\square_i^M U_i)^*(\mu) \geq \square_i^M U_i(X). \end{aligned} \quad (5.30)$$

(ii) \Rightarrow (iii): property (ii) implies (3.12) and thus there exists a $\mu \in \partial \square_i^M U_i(X)$, by Lemma 4.4 and (2.5). Then $\mu \in \mathcal{P}$ by Lemma 4.3, and (2.2) yields

$$\square_i^M U_i(X) = \langle \mu, X \rangle - (\square_i^M U_i)^*(\mu) \geq \sum_{i=1}^n (\langle \mu, \xi_i \rangle - U_i^{M_i^*}(\mu)) \geq \sum_{i=1}^n U_i(\xi_i) = \square_i^M U_i(X). \quad (5.31)$$

Hence $U_i(\xi_i) = \langle \mu, \xi_i \rangle - U_i^{M_i^*}(\mu)$ for all $i = 1, \dots, n$ and $\langle \mu, \xi_0 \rangle = 0$.

(iii) \Leftrightarrow (iv) follows from (2.2).

(iv) \Leftrightarrow (v): it is readily checked that, for $\mu \in \mathcal{P}$,

$$\langle \mu, \xi_0 \rangle = 0 \quad \Leftrightarrow \quad \mu \in \partial U_0(\xi_0) = \{\nu \in \mathcal{P} \mid \langle \nu, \xi_0 \rangle = U_0(\xi_0)\} \text{ and } U_0(\xi_0) = 0.$$

(iv) \Leftrightarrow (vi): follows from the fact that $\partial V_0(\xi_0) = \{\nu \in (L^\infty)_+^* \mid \langle \nu, \xi_0 \rangle = 0\}$ (note that $\partial V_0(\xi) = \emptyset$ for $\xi \notin L_+^\infty$).

Moreover, (5.30) and (5.31) together with (2.2) and the above arguments imply (5.29). \square

Remark 5.2 Property (vi) ((v)) says that markets clear in a Pareto optimal allocation by introducing a dummy agent with utility function V_0 (U_0). In fact, V_0 (U_0) is the most risk averse concave (monetary) utility function, in the sense that every concave (monetary) utility function W with $W(0) = 0$ satisfies $W \geq V_0$ ($W \geq U_0$), see [16] for a proof.

6 σ -additive equilibrium pricing rules

In this section we assume unconstrained consumption, that is, $M_i = L^\infty$ for all $i = 1, \dots, n$. We provide sufficient conditions under which the finitely additive equilibrium pricing rules $\mu \in \mathcal{P}$ are in fact expectations with respect to absolutely continuous probability measures $Q \ll P$. In what follows, we identify $Q \ll P$ with its density $dQ/dP \in \mathcal{P}_\sigma := \{Z \in L^1_+ \mid E[Z] = 1\}$. With the usual embedding $L^1 \subset (L^1)^{**} = (L^\infty)^*$ one then has $\mathcal{P}_\sigma \subset \mathcal{P}$.

We first cite a classical result (for a proof see e.g. [17]) which states that under some additional continuity assumptions on U the representation (A.33) can be improved. See [19] for the law-invariant case.

Theorem 6.1 *For a monetary utility function U with conjugate function U^* the following properties are equivalent:*

- (i) $U(\xi) = \inf_{Z \in \mathcal{P}_\sigma} (E[Z\xi] - U^*(Z)), \quad \forall \xi \in L^\infty$;
- (ii) *the acceptance set $\mathcal{C} = \{\xi \in L^\infty \mid U(\xi) \geq 0\}$ is $\sigma(L^\infty, L^1)$ -closed;*
- (iii) *U has the Fatou property: for every L^∞ -bounded sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ converging in probability to some $\xi \in L^\infty$ we have $U(\xi) \geq \limsup_{n \geq 1} U(\xi_n)$;*
- (iv) *U is continuous from above: for every decreasing sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ which converges to $\xi \in L^\infty$ almost surely, it follows that $U(\xi_n) \rightarrow U(\xi)$.*

The proof of Proposition 2.5 uses that \mathcal{P} is $\sigma((L^\infty)^*, L^\infty)$ -compact and the conjugate function U^* is $\sigma((L^\infty)^*, L^\infty)$ -upper semi-continuous. Hence if we can provide sufficient conditions such that the level sets

$$\mathcal{Q}^K = \{Q \in \mathcal{P}_\sigma \mid U^*(Q) \geq -K\}, \quad K \in \mathbb{N},$$

are $\sigma(L^1, L^\infty)$ -compact and U^* is $\sigma(L^1, L^\infty)$ -upper semi-continuous, then we have existence of a $\mu \in \mathcal{P}_\sigma$ which satisfies (2.3). It turns out that continuity from below of U is enough. This is the content of our next theorem, which is a generalization of Theorem 3.6 in [10] to the non-coherent case, and the proof of which is given in Section C.

Theorem 6.2 *For a monetary utility function U with conjugate function U^* the following properties are equivalent:*

- (i) \mathcal{Q}^K is uniformly integrable for all $K \in \mathbb{N}$;
- (ii) \mathcal{Q}^K is $\sigma(L^1, L^\infty)$ -compact for all $K \in \mathbb{N}$;
- (iii) *for every increasing sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ which converges to 0 almost surely, it follows $U(\xi_n) \rightarrow 0$;*
- (iv) *U is continuous from below: for every increasing sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ which converges to $\xi \in L^\infty$ almost surely, it follows $U(\xi_n) \rightarrow U(\xi)$.*

Any of the above properties implies properties (i)–(iv) of Theorem 6.1.

With the preceding remarks, Theorem 6.2 leads to an improvement of Proposition 2.5 (see [17, Proposition 4.21] for a different proof):

Corollary 6.3 *Assume that any of the properties (i)–(iv) in Theorem 6.2 holds. Then for any $\xi \in L^\infty$ there exists a $Z \in \mathcal{P}_\sigma$ with*

$$U(\xi) = E[Z\xi] - U^*(Z). \quad (6.32)$$

Moreover, $\partial U(\xi) \cap \mathcal{P}_\sigma \neq \emptyset$.

In view of the preceding sections we need to know whether continuity from below of any U_i implies continuity from below of the unconstrained convolution $\square_i U_i$, see (4.22). It turns out that this is indeed the case, see also [3].

Lemma 6.4 *Let U_1, \dots, U_n be monetary utility functions. Suppose that U_1 is continuous from below. Then $\square_i U_i$ is continuous from below.*

Proof. We can assume that $\square_i U_i(0) < \infty$, since otherwise there is nothing to be proved. Consider a sequence $(\xi_n)_{n \in \mathbb{N}} \subset L^\infty$ which increases to $\xi \in L^\infty$. Then,

$$\begin{aligned} \sup_{k \in \mathbb{N}} \square_{i=1}^n U_i(\xi_k) &= \sup_{k \in \mathbb{N}} \sup_{\eta \in L^\infty} (U_1(\xi_k - \eta) + \square_{i=2}^n U_i(\eta)) \\ &= \sup_{\eta \in L^\infty} \sup_{k \in \mathbb{N}} (U_1(\xi_k - \eta) + \square_{i=2}^n U_i(\eta)) \\ &= \sup_{\eta \in L^\infty} (U_1(\xi - \eta) + \square_{i=2}^n U_i(\eta)) = \square_{i=1}^n U_i(\xi). \end{aligned}$$

□

Note, however, that Lemma 6.4 does not hold under constrained consumption in general, even if all U_i are continuous from below, as the following example shows.

Example 6.5 Consider the one-dimensional linear subspaces $M_1 = M_2 = \mathbb{R} \subset L^\infty$. Then, for any monetary utility functions U_1, U_2 and $X \in L^\infty$, we have

$$\square_i^M U_i(X) = \sup_{\xi_1 + \xi_2 \leq X, \xi_i \in \mathbb{R}} (U_1(\xi_1) + U_2(\xi_2)) = \sup_{\xi_1 + \xi_2 \leq \text{ess inf } X, \xi_i \in \mathbb{R}} (\xi_1 + \xi_2) = \text{ess inf } X,$$

which is not continuous from below if (Ω, \mathcal{F}, P) is atomless.

Summarizing Corollary 6.3 and Lemma 6.4 we can now improve Theorem 3.5 as follows:

Theorem 6.6 *Consider the setup of Section 3 under the assumption that $M_i = L^\infty$ for all $i = 1, \dots, n$. Suppose that at least one U_i is continuous from below. Then there exists an equilibrium pricing rule $\mu \in \partial \square_i^M U_i(X) \cap \mathcal{P}_\sigma$ if and only if (3.11) or (3.12) holds.*

A Proof of Proposition 2.5

It follows by inspection that U^M is proper, concave and cash invariant. Since U^M is also monotone on M , we have that

$$|U^M(X) - U^M(Y)| \leq \|X - Y\|_\infty \quad \text{for all } X, Y \in M.$$

Hence, for all $c \in \mathbb{R}$, the set $\{X \in L^\infty \mid U^M(X) \geq c\}$ is L^∞ -closed and, since convex, therefore $\sigma(L^\infty, (L^\infty)^*)$ -closed (see [25, Theorem 3.12]). Hence U^M is $\sigma(L^\infty, (L^\infty)^*)$ -upper semi-continuous.

From the monotonicity and cash invariance of U it follows that

$$U(\xi) = U^{**}(\xi) = \inf_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^*(\mu)), \quad \xi \in L^\infty, \quad (\text{A.33})$$

see e.g. [17]. On the other hand, for every $\mu \in (L^\infty)^*$ we have

$$U^*(\mu) = \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - U(\xi)) \leq \inf_{\xi \in L^\infty} (\langle \mu, \xi \rangle - U^M(\xi)) = U^{M*}(\mu).$$

Combining this with (A.33), we obtain for $\xi \in M$

$$\begin{aligned} U^M(\xi) = U(\xi) &= \inf_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^*(\mu)) \geq \inf_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^{M*}(\mu)) \\ &\geq \inf_{\mu \in (L^\infty)^*} (\langle \mu, \xi \rangle - U^{M*}(\mu)) = U^M(\xi). \end{aligned}$$

This shows that $U^M(\xi) = \inf_{\mu \in \mathcal{P}} (\langle \mu, \xi \rangle - U^{M*}(\mu))$ for all $\xi \in M$. It remains to show that this inf is attained.

Let $\xi \in M$, and write $g(\mu) := \langle \mu, \xi \rangle - U^{M*}(\mu)$. Since $|\langle \mu, \eta \rangle| \leq \langle \mu, 1 \rangle \|\eta\|_\infty = \|\eta\|_\infty$ for all $\eta \in L^\infty$, the Alaoglu Compactness Theorem ([1, Theorem 5.93]) implies that \mathcal{P} is $\sigma((L^\infty)^*, L^\infty)$ -compact. On the other hand, g is $\sigma((L^\infty)^*, L^\infty)$ -lower semi-continuous. Hence, by a generalization of Weierstrass' Theorem ([1, Theorem 2.40]), g attains a minimum on \mathcal{P} . This proves (2.3).

Let $\mu \in \mathcal{P}$. Then $\langle \mu, \eta + \langle \mu, \xi - \eta \rangle \rangle = \langle \mu, \xi \rangle$. Hence

$$\begin{aligned} \sup_{\langle \mu, \eta \rangle \leq \langle \mu, \xi \rangle} U^M(\eta) &= \sup_{\eta \in L^\infty} U^M(\eta + \langle \mu, \xi - \eta \rangle) \\ &= \langle \mu, \xi \rangle - \inf_{\eta \in L^\infty} (\langle \mu, \eta \rangle - U^M(\eta)) = \langle \mu, \xi \rangle - U^{M*}(\mu), \end{aligned}$$

which is (2.4).

Property (2.5) is a consequence of (2.3) and (2.2), and (2.6) follows by inspection.

B Proof of Theorem 3.11

Let $(\xi_1, \dots, \xi_n, \mu)$ be an equilibrium. Then μ is an equilibrium pricing rule and Property (ii) follows from Lemma 4.3. Moreover, $U_i^{M_i}(\xi_i) = U_i(\xi_i) = \sup_{\langle \mu, \zeta \rangle \leq \langle \mu, \xi_i \rangle} U_i^{M_i}(\zeta)$ and

$\langle \mu, \xi_i \rangle = \langle \mu, X_i \rangle$, for all $i = 1, \dots, n$. Indeed, otherwise by cash invariance $\xi_i + \langle \mu, X_i - \xi_i \rangle \in M_i$ would yield a strictly bigger utility than ξ_i . Hence $\langle \mu, \xi_0 \rangle = 0$, for ξ_0 as in Theorem 5.1, and (2.2) implies that $\mu \in \bigcap_{i=1}^n \partial U_i^{M_i}(\xi_i)$. Now Theorem 5.1 yields Pareto optimality of (ξ_1, \dots, ξ_n) .

Conversely, let (ξ_1, \dots, ξ_n) be Pareto optimal and $\mu \in \partial \square_i^M U_i(X)$. Then $\eta_i := \xi_i + \langle \mu, X_i - \xi_i \rangle \in M_i$ and $\xi_0 := X - \sum_{i=1}^n \xi_i$ satisfy, by Theorem 5.1,

$$\sum_{i=1}^n \eta_i = \sum_{i=1}^n \xi_i + \langle \mu, \xi_0 \rangle = \sum_{i=1}^n \xi_i \leq X$$

and $U_i(\eta_i) = U_i(\xi_i) + \langle \mu, X_i - \xi_i \rangle = \langle \mu, X_i \rangle - U_i^{M_i^*}(\mu)$ for all $i = 1, \dots, n$. Hence (η_1, \dots, η_n) is attainable and

$$U_i(\zeta_i) \leq \langle \mu, \zeta_i \rangle - U_i^{M_i^*}(\mu) \leq \langle \mu, X_i \rangle - U_i^{M_i^*}(\mu) = U_i(\eta_i)$$

for all $\zeta_i \in M_i$ with $\langle \mu, \zeta_i \rangle \leq \langle \mu, X_i \rangle$. Therefore $(\eta_1, \dots, \eta_n, \mu)$ is an equilibrium, which proves (3.17).

That (i)–(iii) imply the equilibrium property is a simple consequence of (3.17).

C Proof of Theorem 6.2

(iv) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (iv): Fix $\varepsilon > 0$. Then there exists a $\lambda \in (0, 1)$ such that $\lambda U(\xi) \geq U(\xi) - \varepsilon$ and $\|(1 - \lambda)\xi\|_\infty \leq \varepsilon$. Write $\xi_n = \xi + \eta_n$ for some increasing sequence $(\eta_n)_{n \in \mathbb{N}}$ which tends to 0. For $\tilde{\eta}_n = \frac{\eta_n}{(1 - \lambda)}$ we get $\xi_n = \xi + (1 - \lambda)\tilde{\eta}_n$ and $\|\xi_n - (\lambda\xi + (1 - \lambda)\tilde{\eta}_n)\|_\infty \leq \varepsilon$ and therefore

$$\begin{aligned} U(\xi_n) + \varepsilon &\geq U(\lambda\xi + (1 - \lambda)\tilde{\eta}_n) \\ &\geq \lambda U(\xi) + (1 - \lambda)U(\tilde{\eta}_n) \\ &\geq U(\xi) + (1 - \lambda)U(\tilde{\eta}_n) - \varepsilon. \end{aligned}$$

By assumption $U(\tilde{\eta}_n)$ increases to 0 and the claim follows.

(i) \Leftrightarrow (ii) is a standard result in (L^1, L^∞) duality theory and follows by the Dunford-Pettis theorem.

(iii) \Rightarrow (i): Suppose that \mathcal{Q}^K is not uniformly integrable for some $K \in \mathbb{N}$. Then, there exists a decreasing sequence A_n of elements in \mathcal{F} which converges to \emptyset , such that

$$\limsup_{n \rightarrow \infty} \inf_{Z \in \mathcal{Q}^K} E[-Z1_{A_n}] < 0.$$

There exists a constant $C \in \mathbb{N}$ such that $\limsup_{n \rightarrow \infty} \inf_{Z \in \mathcal{Q}^K} E[-CZ1_{A_n}] \leq -K - 1$. Hence, for the sequence $-C1_{A_n}$ which increases to 0 we get a contradiction by

$$\begin{aligned} U(-C1_{A_n}) &= \inf_{Z \in \mathcal{P}_\sigma} \{E[-CZ1_{A_n}] - U^*(Z)\} \\ &\leq \inf_{Z \in \mathcal{Q}^K} \{E[-CZ1_{A_n}] - U^*(Z)\} \\ &\leq -1. \end{aligned}$$

(i) \Rightarrow (iii): We first assume that $U(0) = 0$. Fix $\varepsilon > 0$. Suppose that $(\xi_n)_{n \in \mathbb{N}}$ increases to zero. For $A_n = \{\omega \in \Omega \mid \xi_n(\omega) < -\varepsilon\}$, it follows that $P[A_n] \rightarrow 0$. Since $(\xi_n)_{n \in \mathbb{N}}$ is L^∞ -bounded, there exists $K \in \mathbb{N}$ such that

$$\begin{aligned} U(\xi_n) &= \inf_{Z \in \mathcal{P}_\sigma} \{E[Z\xi_n] - U^*(Z)\} = \inf_{Z \in \mathcal{Q}^K} \{E[Z\xi_n] - U^*(Z)\} \\ &\geq \inf_{Z \in \mathcal{Q}^K} E[Z(\xi_n 1_{A_n} + \xi_n 1_{A_n^c})] \\ &\geq -\|\xi_0\|_\infty \inf_{Z \in \mathcal{Q}^K} E[Z 1_{A_n}] - \varepsilon. \end{aligned}$$

The first inequality follows, because the conjugate function U^* of a monetary utility function U with $U(0) = 0$ takes only values in $[-\infty, 0]$. By assumption, $\inf_{Z \in \mathcal{Q}^K} E[Z 1_{A_n}]$ tends to zero, whence the claim follows if $U(0) = 0$.

By the cash invariance, every monetary utility function U can be transformed in a monetary utility function $\hat{U}(\xi) = U(\xi) - U(0)$ which satisfies $\hat{U}(0) = 0$. For ξ_n increasing to zero, we deduce

$$U(\xi_n) = \hat{U}(\xi_n) + U(0) \rightarrow \hat{U}(0) + U(0) = U(0).$$

It remains to prove that continuity from below implies continuity from above. But concavity of U implies

$$U(\xi + \eta) + U(\xi - \eta) \leq 2U(\xi) \quad \forall \xi, \eta \in L^\infty.$$

Hence $U(\xi) \leq U(\xi + \eta) \leq 2U(\xi) - U(\xi - \eta)$ for all $\eta \geq 0$, and the assertion follows.

References

- [1] Aliprantis, C.D. and Border, K.C.: Infinite Dimensional Analysis, 2nd Edition. Springer. (1999)
- [2] Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. *Math. Finance* **9**(3), 203–228 (1999)
- [3] Barrieu, P., El Karoui, N.: Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics* **9**, 269–298 (2005).
- [4] Bewley, T.F.: Existence of Equilibria in Economies with Infinitely Many Commodities. *Journal of Economic Theory* **4**, 514–540 (1972)
- [5] Burgert Ch. and L. Rüschendorf: Allocation of risks and equilibrium in markets with finitely many traders. Preprint. (2005)
- [6] Cherny A.S.: Equilibrium with coherent risk. Preprint. (2005)
- [7] Dana R.-A.: Existence, uniqueness and determinacy of equilibrium in C.A.P.M. with a riskless asset. *Journal of Mathematical Economics* **32**, 167–175 (1999)

- [8] Dana R.-A., Jeanblanc M.: *Financial Markets in Continuous Time*. Springer (2003).
- [9] Dana R.-A., Le Van C.: Arbitrage, duality and asset equilibria. *Journal of Mathematical Economics* **34**, 397–413 (2000)
- [10] Delbaen, F.: Coherent risk measures on general probability spaces. *Advances in Finance and Stochastics*, 1–37, Springer, Berlin (2002)
- [11] Delbaen, F.: *Coherent Risk Measures*. Cattedra Galileiana Lecture Notes, Pisa (2002).
- [12] Delbaen, F.: Hedging bounded claims with bounded outcomes. Working Paper, ETH Zürich (2004)
- [13] Dolecki, S., Greco, G. H.: Niveloid, *Topological Methods in Nonlinear Analysis* 5, 1–22 (1995)
- [14] Ekeland, I., Témam, R.: *Convex analysis and variational problems*, Society for Industrial and Applied Mathematics (SIAM) (1999)
- [15] Filipović, D., Kupper, M.: *Optimal Capital and Risk Transfers for Group Diversification*. Forthcoming in *Mathematical Finance*.
- [16] Filipović, D., Kupper, M.: *Monotone and Cash-Invariant Convex Functions and Hulls*. Forthcoming in *Insurance: Mathematics and Economics*.
- [17] Föllmer, H., Schied, A.: *Stochastic Finance, An Introduction in Discrete Time*. de Gruyter Studies in Mathematics **27** (2002)
- [18] Heath D. and H.Ku: Pareto equilibria with coherent measures of risk. *Mathematical Finance* 14, 163-172. (2004)
- [19] Jouini E., Schachermayer W., N. Touzi: Law-invariant risk measures have the Fatou property. Preprint. (2005)
- [20] Jouini E., Schachermayer W., N. Touzi: Optimal risk sharing for law invariant monetary utility functions. Preprint. (2005)
- [21] Maccheroni, F., Marinacci, M., Rustichini, A.: *Ambiguity Aversion, Robustness and the Variational Representation of Preferences*, Working Paper, Bocconi University. (2004)
- [22] Maccheroni, F., Marinacci, M., and Rustichini, A., Taboga, M.: *Portfolio Selection with Monotone Mean-Variance Preferences*. Preprint (2005)
- [23] Rockafellar, R.T.: *Convex Analysis*. Princeton University Press. (1997)
- [24] Rockafellar, R. T., Uryasev, S. and M. Zabarankin.: *Generalized Deviations in Risk Analysis*. *Finance and Stochastics* 10, 51-74 (2006).

- [25] Rudin, W.: Functional Analysis. McGraw-Hill. (1991)
- [26] Werner, J.: Arbitrage and the Existence of Competitive Equilibrium. *Econometrica* **55**, 1403–1418 (1987)