

# Convex risk measures on $L^p$

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## Abstract

Convex risk measures are best known on  $L^\infty$ . In this paper we argue that  $L^p$ , for  $p \in [1, \infty)$ , is a more appropriate model space. We provide a comprehensive but concise exposure of the topological properties of convex risk measures on  $L^p$ . Our main result is the complete characterization of the extension and restriction operations of convex risk measures from and to  $L^\infty$  and  $L^p$ , respectively. In particular, it turns out that there is a one-to-one correspondence between law-invariant convex risk measures on  $L^\infty$  and  $L^1$ .

**Key words:** convex risk measures, extension and restriction operations, law-invariant convex functions

## 1 Introduction

Convex risk measures are best known on  $L^\infty$ , the space of essentially bounded random variables. Indeed, Artzner et al. [2] introduced the seminal axioms of coherence, which then were further generalized to the convex case by Föllmer and Schied [9] and Frittelli and Rosazza-Gianin [11], on  $L^\infty$ .

In this paper we promote  $L^p$ , the space of random variables with finite  $p$ -th moment, for  $p \in [1, \infty)$ , as model space for convex risk measures. There are several reasons which motivate such a study:

- Important risk models, both from a theoretical and applied point of view, such as normal distributed random variables, are contained in  $L^p$  but not in  $L^\infty$ .
- It is often argued that  $L^\infty$  has the advantage of not depending on the particular choice of some probability measure but merely on the equivalence class of null sets. However, the most common risk measures in

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use are law-invariant. Examples are Average Value-at-Risk (example 4.1), entropic risk measure (example 4.2), semi-deviation risk measure (example 4.3), and worst case risk measure (example 4.4). These usually require the full specification of the laws, and thus the probability measure and  $L^p$ .

- The space  $L^p$  carries a locally convex topology, which makes it apt for convex analysis. (The same argument disqualifies the case  $p < 1$ .)

There is a growing mathematical finance literature dealing with the extension of convex risk measures beyond  $L^\infty$ . Delbaen [5] considers as model space  $L^0$ , the space of all random variables. However, the  $L^0$ -topology is not locally convex, which makes it not well going with convex analysis. Most often though, the model space is chosen such that some (generic) risk measure remains finite valued (see e.g. [3, 13] and many others). That is, the model space is essentially determined by the choice of some particular risk measure. But when it comes to e.g. comparisons of a set of risk measures, or optimization problems (see e.g. [7, 8, 12], it is actually more appropriate to fix a simple basic model space, such as  $L^p$ , independently of the risk measures.

It is well known that convex risk measures on  $L^\infty$  are, a fortiori, finite valued and Lipschitz-continuous. In our setting this is no longer true. Convex risk measures on  $L^p$  can be very irregular (examples 4.5, 4.6, 4.8). We will thus restrict to those which are proper (that is,  $(-\infty, \infty]$ -valued) and lower semi-continuous. Such convex functions are perfectly apt for convex duality, and this is what we exploit here.

The main questions that now arise are:

- (i) Can every convex risk measure on  $L^\infty$  be extended to  $L^p$ ? And how?
- (ii) Conversely, is the restriction to  $L^\infty$  of a convex risk measure on  $L^p$  a convex risk measure on  $L^\infty$ ?
- (iii) Do these extension and restriction operations commute?
- (iv) In particular, is there a one-to-one correspondence between (certain classes of) convex risk measures on  $L^\infty$  and  $L^p$ , respectively?

Addressing question (i), we formally define two extensions, one via convex duality and the other via closing the acceptance set. We then give necessary and sufficient conditions for both extensions to be well defined, and if so, show that they coincide (theorem 3.1). The validity of statement (ii) follows by our very definition of convex risk measures. As for question (iii), we illustrate that these operations do not commute in general (example 4.7). We then give necessary and sufficient conditions under which they do commute (theorem 3.2). A major result is that law-invariant convex risk measures always satisfy these conditions (theorem 3.4). In particular, there is a one-to-one correspondence between law-invariant convex risk measures on  $L^\infty$  and  $L^1$ . That is, every law-invariant convex risk measure on  $L^\infty$  is the restriction of a unique law-invariant convex

risk measure on  $L^1$ . This responds to question (iv). In this sense,  $L^1$  is the canonical model space for law-invariant convex risk measures.

In sum, the main contribution of this paper is a comprehensive exposure of convex risk measures on  $L^p$ , with focus on law-invariance, and the complete response to questions (i)–(iv) above. Many of our results actually hold without the monotonicity axiom (remark 3.5). A fact that will prove most useful for future research (see e.g. [8]).

The remainder of the paper is as follows. Section 2 contains the formal setup and the definition of convex risk measures on  $L^p$ . We then investigate the interplay between lower semi-continuity and some topological properties of the acceptance set. We also provide necessary and sufficient conditions for continuity of convex risk measures on  $L^p$ . Many results in this section look seemingly familiar, but require non-straightforward proofs. We believe that such comprehensive but concise exposure of the topological properties of convex risk measures on  $L^p$  is a useful contribution to the mathematical finance literature. Section 3 contains our main results, responding to questions (i)–(iv) above. In section 4, we provide a variety of examples which illustrate the pitfalls with convex risk measures on  $L^p$ . We suppose the reader is familiar with basic duality theory for convex functions as outlined in [14] or [6]. In section A, we give a brief summary of notational conventions and results from convex analysis which will be used throughout the text. Section B is a self-contained exposure of (neither necessarily cash-invariant nor monotone) law-invariant convex functions on  $L^p$ . These results are of interest on their own and form the basis for the proof of our main result on law-invariant convex risk measures (theorem 3.4).

## 2 Convex Risk Measures on $L^p$

Throughout this paper  $(\Omega, \mathcal{F}, \mathbb{P})$  denotes an atom-less probability space, i.e. a probability space supporting a random variable with continuous distribution. All equalities and inequalities between random variables are understood in the  $\mathbb{P}$ -almost sure (a.s.) sense. If not specified otherwise, in the sequel, we let  $p \in [1, \infty]$ , and write  $L^p = L^p(\Omega, \mathcal{F}, \mathbb{P})$  and  $\|\cdot\|_p = \|\cdot\|_{L^p}$ . The topological dual space of  $L^p$  is denoted by  $(L^p)^*$ . It is well known that  $(L^p)^* = L^q$  with  $q = \frac{p}{p-1}$  for  $p < \infty$ , and that  $(L^\infty)^* \supset L^1$  can be identified with  $ba$ , the space of all bounded finitely additive measures on  $(\Omega, \mathcal{F})$  which are absolutely continuous with respect to  $\mathbb{P}$ . With some facilitating abuse of notation, we shall write  $(X, Z) \mapsto E[XZ]$  for the dual pairing on  $(L^p, (L^p)^*)$  also for the case  $p = \infty$ .

**Definition 2.1.** *A function  $\rho : L^p \rightarrow (-\infty, \infty]$  is a convex risk measure on  $L^p$  if it exhibits the following characteristics:*

- (i)  $\rho(0) < \infty$ ,
- (ii) convexity:  $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$  for all  $\lambda \in [0, 1]$ ,
- (iii) cash-invariance:  $\rho(X + r) = \rho(X) - r$  for all  $r \in \mathbb{R}$ ,

(iv) monotonicity:  $X \geq Y$  implies  $\rho(X) \leq \rho(Y)$ .

If in addition  $\rho(tX) = t\rho(X)$  for all  $t \geq 0$ , we call  $\rho$  a coherent risk measure. The acceptance set of a convex risk measure  $\rho$  on  $L^p$  is

$$\mathcal{A}_\rho := \{X \in L^p \mid \rho(X) \leq 0\}.$$

The convex set of “generalized scenarios” (e.g. [2])

$$\mathcal{P}_p := \{Z \in (L^p)^*_- \mid E[Z] = -1\}$$

will be used throughout the text (with the obvious interpretation  $E[Z] = E[1 \cdot Z]$  for  $p = \infty$ ).

The following lemma states some fundamental properties of convex risk measures on  $L^p$  and their acceptance sets. In particular, any convex risk measure  $\rho$  on  $L^p$  admits a continuous affine minorant, i.e. its conjugate  $\rho^*$  is proper.

**Lemma 2.2.** *Let  $\rho$  be a convex risk measure on  $L^p$ , then:*

(i)  $\mathcal{A} = \mathcal{A}_\rho$  is convex and satisfies the following conditions:

$$\begin{aligned} \mathbb{R} \cap \mathcal{A} \neq \emptyset, \quad \overline{\mathcal{A}} \neq L^p, \quad \text{and} \\ X \in \mathcal{A}, Y \in L^p, Y \geq X \Rightarrow Y \in \mathcal{A}. \end{aligned} \tag{2.1}$$

(ii)  $\rho^*$  is proper and  $\text{dom } \rho^* \subset \mathcal{P}_p$ . Moreover, for all  $Z \in \mathcal{P}_p$ :

$$\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[XZ].$$

(iii) If  $\rho$  is coherent, then  $\text{dom } \rho^*$  is closed and  $\rho^* = \delta(\cdot \mid \text{dom } \rho^*)$ .

*Proof.* (i): The convexity of  $\mathcal{A}_\rho$ , and the first and third condition of (2.1) are obvious by  $\rho(0) < \infty$ , cash-invariance, and monotonicity. In order to prove the second condition, we assume for the moment that  $\overline{\mathcal{A}_\rho} = L^p$ . Then for every  $n \in \mathbb{N}$  there is a  $X_n \in \mathcal{A}_\rho$  such that

$$\|X_n - (-n)\|_p = \|X_n + n\|_p \leq \frac{1}{2^n}.$$

By monotonicity we may assume that  $X_n + n \geq 0$  (because  $(X_n \vee -n) \geq X_n$  implies  $(X_n \vee -n) \in \mathcal{A}_\rho$  and  $\|(X_n \vee -n) + n\|_p \leq \|X_n + n\|_p$ ). Let  $Y_n := \sum_{k=1}^n X_k + k$ ,  $n \in \mathbb{N}$ .  $(Y_n)_{n \in \mathbb{N}}$  is a  $L^p$ -Cauchy sequence converging to

$$Y := \left( \sum_{k=1}^{\infty} X_k + k \right) \in L^p, \quad \|Y\|_p \leq 1.$$

Clearly,  $Y \geq Y_n \geq X_n + n$  for all  $n \in \mathbb{N}$ . Thus, by monotonicity, cash-invariance, and  $X_n \in \mathcal{A}_\rho$ :

$$\rho(Y) \leq \rho(X_n + n) \leq \rho(X_n) - n \leq -n \quad \text{for all } n \in \mathbb{N}.$$

Consequently  $\rho(Y) = -\infty$ . But this is a contradiction to the properness of  $\rho$ . Hence,  $\overline{\mathcal{A}_\rho} \subsetneq L^p$ .

(ii): The fact that  $\text{dom } \rho^*$  must be a subset of  $\mathcal{P}_\rho$  is proved in [7] lemma 3.2. The representation of  $\rho^*(Z)$  for any  $Z \in \mathcal{P}_\rho$  is easily verified in view of  $E[Z] = -1$  and cash-invariance of  $\rho$ . We proceed to proving the properness of  $\rho^*$ : Since, according to (i),  $\overline{\mathcal{A}_\rho} \neq L^p$ , there is  $Y \in L^p \setminus \overline{\mathcal{A}_\rho}$ . The Hahn-Banach separating hyperplane theorem ensures the existence of a nontrivial  $Z \in (L^p)^*$  such that  $E[XZ] < E[YZ]$  for all  $X \in \mathcal{A}_\rho$ . For any  $X \in L^p_+$  and all  $t \geq 0$  we have that  $tX + \rho(0) \in \mathcal{A}_\rho$ . Thus,

$$tE[XZ] + \rho(0)E[Z] < E[YZ] \quad \text{for all } X \in L^p_+ \text{ and } t \geq 0,$$

which implies  $E[XZ] \leq 0$  for all  $X \in L^p_+$ , i.e.  $Z \in (L^p)_-^*$ . Suppose  $E[Z] = 0$ . Then,  $E[rZ] = 0$  for all  $r \in \mathbb{R}$  and thus  $E[XZ] = 0$  for all  $X \in L^\infty$ , because  $X + \|X\|_\infty \geq 0$  and  $\|X\|_\infty - X \geq 0$ , and hence  $E[(X + \|X\|_\infty)Z] \leq 0$  and  $E[(\|X\|_\infty - X)Z] \leq 0$ . As  $L^\infty$  is dense in every  $L^p$ , we conclude that  $Z \equiv 0$  which is a contradiction. Therefore,  $E[Z] < 0$ , and by normalising with a constant if necessary we may assume that  $E[Z] = -1$ , so  $Z \in \mathcal{P}_\rho$ . Since

$$\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[XZ] \leq E[YZ] < \infty,$$

we have  $Z \in \text{dom } \rho^*$ . Hence,  $\rho^*$  is proper.

(iii): Coherence implies  $\rho(0) = 0$ , and consequently  $\rho^*(Z) \geq 0$  for all  $Z \in (L^p)^*$ . Moreover, for all  $t > 0$ :

$$\rho^*(Z) = \sup_{X \in L^p} E[ZtX] - \rho(tX) = \sup_{X \in L^p} t \cdot (E[ZX] - \rho(X)) = t \cdot \rho^*(Z)$$

which implies  $\rho^*(Z) \in \{0, \infty\}$ . Hence,  $\rho^* = \delta(\cdot | \text{dom } \rho^*)$ , and  $\rho^*$  being l.s.c. by definition, we have that  $\text{dom } \rho^* = \{Z \in (L^p)^* \mid \rho^*(Z) \leq 0\}$  is closed.  $\square$

It is common knowledge that all convex risk measures on  $L^\infty$  are 1-Lipschitz-continuous. This is easily derived using the cash-invariance and monotonicity properties (see e.g. [10] and remark 2.8 below). However, for  $p \in [1, \infty)$ , convex risk measures on  $L^p$  need not be continuous. Well-known convex risk measures such as the entropic risk measure or the worst case risk measure on  $L^p$  are lower semi-continuous (l.s.c.) but not continuous (see examples 4.2, 4.3 and 4.4). Moreover, there are convex risk measures on  $L^p$  which are not even l.s.c. (see examples 4.5 and 4.8).

Our main results will require that the convex risk measures studied be l.s.c. This is a very useful property, because, according to (A.6), any l.s.c. convex risk measure  $\rho$  on  $L^p$  satisfies  $\rho^{**} = \rho$ . Since, by cash-invariance,  $\rho$  is l.s.c. if and only if  $\mathcal{A}_\rho$  is closed, we obtain:

**Lemma 2.3.** *Let  $\rho$  be a convex risk measure on  $L^p$ . Equivalent are:*

- (i)  $\rho$  is l.s.c.
- (ii)  $\mathcal{A}_\rho$  is closed.

(iii)  $\rho \equiv \rho^{**}$ .

When constructing convex risk measures, a natural approach is to fix a set  $\mathcal{A} \subset L^p$  of acceptable positions which then induces a function on  $L^p$  via

$$\rho_{\mathcal{A}} : L^p \rightarrow [-\infty, +\infty], \quad X \mapsto \inf\{a \in \mathbb{R} \mid X + a \in \mathcal{A}\} \quad (\inf \emptyset := \infty).$$

**Lemma 2.4.** *Let  $\mathcal{A} \subset L^p$  be a convex set satisfying the conditions (2.1). Then,*

- (i)  $\rho_{\mathcal{A}}$  is a convex risk measure on  $L^p$ .
- (ii)  $\rho_{\overline{\mathcal{A}}}$  is a l.s.c. convex risk measure on  $L^p$  and  $\mathcal{A}_{\rho_{\overline{\mathcal{A}}}} = \overline{\mathcal{A}}$ .
- (iii)  $\text{cl}(\rho_{\mathcal{A}}) = \rho_{\overline{\mathcal{A}}}$ .

*Proof.* (i) and (ii): It is obvious that  $\rho_{\mathcal{A}}$  is a convex, cash-invariant, and monotone function such that  $\rho_{\mathcal{A}}(0) < \infty$ . In order to verify that  $\rho_{\mathcal{A}}$  is also proper, it suffices to show that  $\rho_{\overline{\mathcal{A}}}$  is proper because  $\rho_{\overline{\mathcal{A}}} \leq \rho_{\mathcal{A}}$ . Observe that  $\overline{\mathcal{A}}$  is convex and satisfies the conditions (2.1) because  $\mathcal{A}$  does. Hence,  $\rho_{\overline{\mathcal{A}}}$  is convex, cash-invariant, and monotone too, and  $\rho_{\overline{\mathcal{A}}}(0) < \infty$ . If we had  $\rho_{\overline{\mathcal{A}}}(0) = -\infty$ , then, applying the conditions (2.1), it follows that  $\mathbb{R} \subset \overline{\mathcal{A}}$ , and thus  $L^\infty \subset \overline{\mathcal{A}}$ , so actually  $\overline{\mathcal{A}} = L^p$  which is a contradiction to  $\overline{\mathcal{A}} \neq L^p$ . Consequently,  $\rho_{\overline{\mathcal{A}}}(0) > -\infty$ . Clearly,  $\overline{\mathcal{A}} = \{X \in L^p \mid \rho_{\overline{\mathcal{A}}}(X) \leq 0\}$ . In conjunction with the cash-invariance of  $\rho_{\overline{\mathcal{A}}}$ , this implies that  $\rho_{\overline{\mathcal{A}}}$  is l.s.c. Since any l.s.c. convex function which assumes the value  $-\infty$  cannot take any finite value (see [6] proposition 2.4.), and since  $\rho_{\overline{\mathcal{A}}}(0) \in \mathbb{R}$ , we conclude that  $\rho_{\overline{\mathcal{A}}}$  is proper.

(iii): It is easily verified that  $\mathcal{A} \subset \mathcal{A}_{\rho_{\mathcal{A}}} \subset \overline{\mathcal{A}}$ . In view of (i), (ii) and lemma 2.2 we obtain for all  $Z \in \mathcal{P}_p$  that

$$\rho_{\overline{\mathcal{A}}}^*(Z) = \sup_{X \in \overline{\mathcal{A}}} E[XZ] = \sup_{X \in \mathcal{A}_{\rho_{\mathcal{A}}}} E[XZ] = \rho_{\mathcal{A}}^*(Z).$$

Hence, by lemma 2.3 and (A.6) we conclude that  $\rho_{\overline{\mathcal{A}}} = \rho_{\overline{\mathcal{A}}}^{**} = \rho_{\mathcal{A}}^{**} = \text{cl}(\rho_{\mathcal{A}})$ .  $\square$

**Corollary 2.5.** *For any convex risk measure  $\rho$  on  $L^p$  we have  $\text{cl}(\rho) = \rho_{\overline{\mathcal{A}_\rho}}$ . In particular,  $\text{cl}(\rho)$  is a l.s.c. convex risk measure on  $L^p$  with acceptance set  $\overline{\mathcal{A}_\rho}$ .*

*Proof.* Since  $\rho = \rho_{\mathcal{A}_\rho}$ , this is an immediate consequence of lemma 2.4.  $\square$

We now turn to a characterisation of continuous convex risk measures.

**Lemma 2.6.** *Let  $\rho$  be a convex risk measure on  $L^p$ . The following conditions are equivalent:*

- (i)  $\rho$  is continuous.
- (ii)  $\rho$  is continuous at one point  $X \in \text{dom } \rho$ .
- (iii)  $\text{int } \mathcal{A}_\rho \neq \emptyset$ .
- (iv)  $\rho$  is l.s.c. and  $\text{dom } \rho = L^p$ .

In either case  $\rho(X) = \max_{Z \in \mathcal{P}_\rho} E[XZ] - \rho^*(Z)$ .

*Proof.* (i)  $\Rightarrow$  (ii): trivial.

(ii)  $\Rightarrow$  (iii): Let  $\epsilon > 0$ . Then, there is an open neighbourhood  $A$  of  $X$  such that  $\rho(A) \subset (\rho(X) - \epsilon, \rho(X) + \epsilon)$ , and by cash-invariance  $A + \rho(X) + \epsilon \subset \mathcal{A}_\rho$ .

(iii)  $\Rightarrow$  (iv): We claim that  $\text{dom } \rho = L^p$ : Assume there is a  $\tilde{X} \in L^p$  such that  $\rho(\tilde{X}) = \infty$ . Since the interior of the convex set  $\text{dom } \rho$  is not empty and  $\tilde{X} \notin \text{dom } \rho$ , an appropriate version of the Hahn-Banach separating hyperplane theorem ensures the existence of a nontrivial  $Z \in (L^p)^*$  such that

$$\sup_{Y \in \text{dom } \rho} E[ZY] \leq E[Z\tilde{X}].$$

Repeating the arguments from the proof of lemma 2.2 (ii), we conclude that  $Z$  must satisfy  $Z \in (L^p)_-$  and  $E[Z] < 0$ . Clearly,  $Y := r/E[Z] \in \text{dom } \rho$  for all  $r \geq 0$ . Hence,  $E[Z\tilde{X}] \geq r$  for all  $r \geq 0$  which is a contradiction. Therefore, such a  $Z$  cannot exist, so  $\text{dom } \rho = L^p$ . As  $\rho$  is bounded from above by 0 on  $\text{int } \mathcal{A}_\rho$ , it is continuous (and thus l.s.c.) over the interior of its domain (see [6] proposition 2.5) which we have shown is  $L^p$ .

(iv)  $\Rightarrow$  (i): Any l.s.c. convex function on a Banach space is continuous over the interior of its domain (see [6] corollary 2.5).

Finally, according to [6] proposition 5.2, the subgradient of a finite continuous convex function is non-empty at each point. Hence, the stated representation of  $\rho$  follows by (A.7).  $\square$

The following lemma characterises continuity of a l.s.c. convex risk measure in terms of the domain of its conjugate function.

**Lemma 2.7.** *Let  $\rho$  be a l.s.c. convex risk measure on  $L^p$ .*

- (i) *If  $\text{dom } \rho^*$  is relatively compact w.r.t. the  $\sigma((L^p)^*, L^p)$  topology, then  $\rho$  is continuous.*
- (ii) *If  $\rho$  is coherent, then  $\text{dom } \rho^*$  is  $\sigma((L^p)^*, L^p)$ -compact if and only if  $\rho$  is continuous.*

*Proof.* (i): Suppose  $\overline{\text{dom } \rho^*}$  is compact. For all  $X \in L^p$  the functions  $g_X(Z) := E[XZ] - \rho^*(Z)$ ,  $Z \in (L^p)^*$ , are upper semi-continuous. Hence, by a generalised version of Weierstrass' theorem ([1] theorem 2.40),  $g_X$  attains its maximum on  $\overline{\text{dom } \rho^*}$ . Clearly, any such maximiser  $Z_X$  is an element of  $\text{dom } \rho^*$ . Therefore,

$$\rho(X) = \rho^{**}(X) = \max_{Z \in \text{dom } \rho^*} g_X(Z) = E[XZ_X] - \rho^*(Z_X).$$

In particular,  $\rho$  takes finite values only. Thus we know from lemma 2.6 that  $\rho$  must be continuous.

(ii): Suppose that  $\rho$  is coherent and continuous. Then

$$\rho(X) = \max_{Z \in \text{dom } \rho^*} E[XZ] \quad \text{for all } X \in L^p.$$

This implies

$$\sup_{Z \in \text{dom } \rho^*} |E[XZ]| < \infty \quad \text{for all } X \in L^p.$$

Hence, applying the Banach-Steinhaus theorem, we deduce that the closed set  $\text{dom } \rho^*$  is bounded, and thus  $\sigma((L^p)^*, L^p)$ -compact due to the Alaoglu theorem.  $\square$

The gist of lemma 2.7 is that every continuous coherent risk measure is a support function over a weak\*-compact subset of  $\mathcal{P}_p$ , and vice versa. As a simple application of this result, we will derive in example 4.1 that the Average Value at Risk is a continuous coherent risk measure on  $L^p$ .

**Remark 2.8.** Note that for all  $Z \in \mathcal{P}_\infty$  we have that

$$\sup_{X \in L^\infty, \|X\|_\infty = 1} |E[XZ]| = 1.$$

Hence, the closed set  $\mathcal{P}_\infty$  is bounded and thus  $\sigma(ba, L^\infty)$ -compact.

### 3 Extensions of Convex Risk Measures

This section contains our main results responding to questions (i)–(iv) in the introduction. Throughout this section, we let  $1 \leq p < r \leq \infty$ .

First, let  $\rho$  be a convex risk measure on  $L^r$ . Our aim is to extend  $\rho$  to a convex risk measure on  $L^p$ . By convex duality, the formal candidate is

$$\rho^p(X) := \sup_{Z \in \mathcal{P}_p} E[ZX] - \rho^*(Z), \quad X \in L^p. \quad (3.2)$$

However,  $\rho^p$  may become  $-\infty$  (see example 4.6). In view of lemma 2.4, another approach to the extension problem is to interpret  $\mathcal{A}_\rho$  as a subset of  $L^p$  and to look at  $\rho_{\overline{\mathcal{A}_\rho^p}}$  where  $\overline{\mathcal{A}_\rho^p}$  denotes the  $\|\cdot\|_p$ -closure of  $\mathcal{A}_\rho$ . Our first main result shows that these two notions of extension coincide:

**Theorem 3.1.** *Let  $\rho$  be a convex risk measure on  $L^r$ . Equivalent are:*

- (i)  $\rho^p$  is a convex risk measure on  $L^p$ .
- (ii)  $\rho^p$  is a l.s.c. convex risk measure on  $L^p$ .
- (iii)  $\text{dom } \rho^* \cap (L^p)^* \neq \emptyset$ .
- (iv)  $\overline{\mathcal{A}_\rho^p} \neq L^p$ .

*In either case  $(\rho^p)^* = \rho^*|_{(L^p)^*}$  and  $\rho^p = \rho_{\overline{\mathcal{A}_\rho^p}}$ .*

*Proof.* (i)  $\Leftrightarrow$  (ii): Obvious by definition of  $\rho^p$ .

(ii)  $\Leftrightarrow$  (iii): If (iii) does not hold, then  $\rho^p \equiv -\infty$ , so it cannot be a convex risk measure. On the other hand, if (iii) does hold, it is easily verified that  $\rho^p$  is a l.s.c. convex risk measure.

(ii)  $\Rightarrow$  (iv): Clearly,  $\rho^p \leq \rho$  on  $L^r$ . Hence, we have that  $\mathcal{A}_\rho \subset \mathcal{A}_{\rho^p}$ . Now  $\mathcal{A}_{\rho^p}$  being closed ensures that  $\overline{\mathcal{A}}_\rho^p \subset \mathcal{A}_{\rho^p}$ . Assume for the moment that there is  $Y \in \mathcal{A}_{\rho^p} \setminus \overline{\mathcal{A}}_\rho^p$ . Then, by the Hahn-Banach separation theorem, there is a  $\tilde{Z} \in (L^p)^*$  such that

$$\sup_{X \in \overline{\mathcal{A}}_\rho^p} E[\tilde{Z}X] < E[\tilde{Z}Y].$$

By the same arguments as in the proof of lemma 2.2 (ii), we may assume that  $\tilde{Z} \in \mathcal{P}_p$ . Hence,  $\rho^*(\tilde{Z}) = \sup_{X \in \mathcal{A}_\rho} E[\tilde{Z}X] < E[\tilde{Z}Y]$  and thus  $\tilde{Z} \in \text{dom } \rho^* \cap (L^p)^*$ . Consequently,

$$\rho^p(Y) \geq E[Y\tilde{Z}] - \rho^*(\tilde{Z}) > E[Y\tilde{Z}] - E[Y\tilde{Z}] = 0$$

which is a contradiction to  $Y \in \mathcal{A}_{\rho^p}$ . Therefore,  $\overline{\mathcal{A}}_\rho^p = \mathcal{A}_{\rho^p}$ , so  $\overline{\mathcal{A}}_\rho^p$  must satisfy conditions (2.1).

(iv)  $\Rightarrow$  (ii) : Clearly,  $\overline{\mathcal{A}}_\rho^p \cap \mathbb{R} \neq \emptyset$ , and we have  $\overline{\mathcal{A}}_\rho^p \neq L^p$  by assumption. Let  $X \in \overline{\mathcal{A}}_\rho^p$  and  $Y \in L^p$  such that  $Y \geq X$ . Choose  $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}_\rho$  converging to  $X$  in  $(L^p, \|\cdot\|_p)$ . We have  $(Y - X) \wedge n \in L^p_+$  for all  $n \in \mathbb{N}$ , and thus  $Y_n := (Y - X) \wedge n + X_n \in \mathcal{A}_\rho$  for all  $n \in \mathbb{N}$ . Since  $Y_n$  converges to  $Y$  w.r.t. the  $\|\cdot\|_p$ -norm, we conclude that  $Y \in \overline{\mathcal{A}}_\rho^p$ . Consequently,  $\overline{\mathcal{A}}_\rho^p$  satisfies the conditions (2.1). Hence, by lemma 2.4,  $\rho_{\overline{\mathcal{A}}_\rho^p}^p$  is a l.s.c. convex risk measure on  $L^p$ . For all  $Z \in \mathcal{P}_p$  we have that

$$\rho_{\overline{\mathcal{A}}_\rho^p}^*(Z) = \sup_{X \in \overline{\mathcal{A}}_\rho^p} E[XZ] = \sup_{X \in \mathcal{A}_\rho} E[XZ] = \rho^*(Z).$$

Hence,  $\rho_{\overline{\mathcal{A}}_\rho^p}^* = \rho^*|_{(L^p)^*}$ . Therefore,  $\rho_{\overline{\mathcal{A}}_\rho^p} = \rho_{\overline{\mathcal{A}}_\rho^p}^{**} = \rho^p$ . Thus (ii) and the closing statements of the theorem follow.  $\square$

On the other hand, any convex risk measure  $\rho$  on  $L^p$  induces a convex risk measure on  $L^r$  by restriction:  $\rho_r := \rho|_{L^r}$ . Under what conditions do the extension and restriction operations,  $\rho \mapsto \rho^p$  and  $\rho \mapsto \rho_r$ , respectively, commute?

**Theorem 3.2.** *Let  $\rho$  be a convex risk measure on  $L^r$ . Then the following conditions are equivalent:*

- (i)  $\rho$  is  $\sigma(L^r, (L^p)^*)$ -l.s.c.
- (ii)  $(\rho^p)_r = \rho$ .

*Conversely, let  $\rho$  be a l.s.c. convex risk measure on  $L^p$ . Then the following conditions are equivalent:*

- (iii)  $\mathcal{A}_{\rho_r} = \mathcal{A}_\rho \cap L^r$  is dense in  $\mathcal{A}_\rho$ .

(iv)  $\rho^* = (\rho_r)^*|_{(L^p)^*}$ .

(v)  $(\rho_r)^p = \rho$ .

*Proof.* (i)  $\Leftrightarrow$  (ii): Clearly,  $(\rho^p)_r = \rho$  is equivalent to

$$\forall X \in L^r : \quad \rho(X) = \sup_{Z \in \mathcal{P}_p} E[XZ] - \rho^*(Z),$$

which is equivalent to the  $\sigma(L^r, (L^p)^*)$ -l.s.c. of  $\rho$ .

(iii)  $\Rightarrow$  (iv): For all  $Z \in \mathcal{P}_p$  we have

$$\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[XZ] = \sup_{X \in \mathcal{A}_{\rho_r}} E[XZ] = (\rho_r)^*(Z).$$

(iv)  $\Rightarrow$  (v): is obvious.

(v)  $\Rightarrow$  (iii): Applying theorem 3.1 we obtain  $\rho = (\rho_r)^p = \rho_{\overline{\mathcal{A}}_{\rho_r}^p}$ , so  $\mathcal{A}_\rho = \overline{\mathcal{A}}_{\rho_r}^p$ .  $\square$

In general the extension and restriction operations do not commute (see example 4.7). However, as we shall see now, in case of law-invariance they do.

**Definition 3.3.** We write  $X \sim Y$  if the random variables  $X$  and  $Y$  have the same law. A function  $f : L^p \rightarrow [-\infty, \infty]$  is called law-invariant if  $f(X) = f(Y)$  whenever  $X \sim Y$ .

**Theorem 3.4.** (i) If  $\rho$  is a l.s.c. law-invariant convex risk measure on  $L^r$ , then  $\rho^p$  is law-invariant, and  $(\rho^p)_r = \rho$ .

(ii) If  $\rho$  is a l.s.c. law-invariant convex risk measure on  $L^p$ , then  $\rho_r$  is law-invariant, and  $(\rho_r)^p = \rho$ .

In particular, every l.s.c. law-invariant convex risk measure  $\rho$  on  $L^r$  is the restriction to  $L^r$  of a unique l.s.c. law-invariant convex risk measure on  $L^1$ . Hence

$$\rho(X) = \sup_{Z \in \mathcal{P}_1} E[XZ] - \rho^*(Z), \quad X \in L^r, \quad (3.3)$$

i.e.  $\rho$  is  $\sigma(L^r, L^\infty)$ -l.s.c.

*Proof.* (i): The assertion  $(\rho^p)_r = \rho$  is proved by lemma B.2 and theorem 3.2, whereas the law-invariance of  $\rho^p$  is a consequence of lemma B.6 in combination with  $(\rho^p)^* = \rho^*|_{(L^p)^*}$  (see theorem 3.1).

(ii): The law-invariance of  $\rho_r$  is obvious. For any  $X \in L^p$  there exists a finite partition  $A_1, \dots, A_n$  of  $\Omega$  such that the distance between  $X$  and the simple random variable  $X_m := E[X | \sigma(A_1, \dots, A_n)] \in L^\infty \subset L^r$ ,  $m \in \mathbb{N}$ , is less than  $1/m$ . On the one hand, lemma B.8 (v) implies that  $\rho(X_m) \leq \rho(X)$  for all  $m \in \mathbb{N}$ . On the other hand, according to the l.s.c. of  $\rho$ , we know that  $\rho(X) \leq \liminf_{m \rightarrow \infty} \rho(X_m)$ . Hence,  $\rho(X) = \lim_{m \rightarrow \infty} \rho(X_m)$ . Therefore, for any  $Z \in (L^p)^*$ , if  $X \in \text{dom } \rho$ , then for arbitrary  $\epsilon > 0$  there is  $m \in \mathbb{N}$  such that  $|(E[XZ] - \rho(X)) - (E[X_m Z] - \rho(X_m))| < \epsilon$ . Since  $L^\infty \subset L^r$  we have that

$$\rho^*(Z) = \sup_{X \in L^p} E[XZ] - \rho(X) = \sup_{X \in L^r} E[XZ] - \rho(X) = (\rho_r)^*(Z).$$

Now apply theorem 3.2 (iv).

The closing assertions of the theorem are a direct consequence of (i) and (ii) applied to  $p = 1$ .  $\square$

**Remark 3.5.** Let  $f : L^r \rightarrow (-\infty, \infty]$  be a proper l.s.c. convex function, and define, in analogy to (3.2),

$$f^p(X) := \sup_{Z \in (L^p)^*} E[XZ] - f^*(Z), \quad X \in L^p.$$

After verifying that indeed  $(f^p)^* = f^*|_{(L^p)^*}$ , and that the implications (i)  $\Rightarrow$  (ii) and (iv)  $\Rightarrow$  (v) of theorem 3.2 are true for all proper l.s.c. convex functions on  $L^r$  and  $L^p$  respectively, we realise that the proof of theorem 3.4 neither relies on the monotonicity nor on the cash-invariance property of  $\rho$ . In fact, the assertions (i) and (ii) of the theorem hold for any proper l.s.c. law-invariant convex functions on  $L^r$  and  $L^p$  respectively, and thus the representation result (3.3) is true for every proper l.s.c. law-invariant convex function on  $L^r$ , if we replace  $\mathcal{P}_1$  by  $L^\infty$ .

## 4 Examples

The first example of this section is the extension of the well-known Average Value at Risk to any  $L^p$ . Applying lemma 2.7, we will show that it is a continuous coherent risk measure on  $L^p$ .

**Example 4.1.** Average Value at Risk: Let  $\alpha \in (0, 1]$  and

$$\mathcal{Z} := \{Z \in \mathcal{P}_1 \mid Z \geq -\frac{1}{\alpha}\}.$$

According to the Alaoglu theorem,  $\mathcal{Z}$  is  $\sigma((L^p)^*, L^p)$ -compact. Hence, by lemma 2.7 (ii), the corresponding support function

$$\text{AVaR}_\alpha(X) := \max_{Z \in \mathcal{Z}} E[XZ], \quad X \in L^p,$$

is a continuous coherent risk measure. Since  $\text{AVaR}_\alpha^* = \delta(\cdot \mid \mathcal{Z})$  is law-invariant, we deduce by means of lemma B.6 that  $\text{AVaR}_\alpha$  must be law-invariant. Moreover, we claim that

$$\text{AVaR}_\alpha(X) = -\frac{1}{\alpha} \int_0^\alpha q_X(s) ds, \quad X \in L^p, \quad (4.4)$$

where  $q_X$  denotes the (left-continuous) quantile function of  $X$  (see (B.8)). From definition 4.43 and theorem 4.47 in [10], we know that (4.4) holds for all  $X \in L^\infty$ . Now, let  $X \in L^p$ . We approximate  $X$  in  $(L^p, \|\cdot\|_p)$  by a sequence  $(X_n)_{n \in \mathbb{N}}$  of simple random variables such that  $X_n^+ \leq X^+$  and  $X_n^- \leq X^-$ ,  $n \in \mathbb{N}$ . Then, in view of (B.9),  $0 \leq (q_{X_n})^{+/-} \leq (q_X)^{+/-}$ , and hence  $|q_{X_n}| \leq |q_X|$  for all  $n \in \mathbb{N}$ . Furthermore, since  $q_X = \lim_{n \rightarrow \infty} q_{X_n}$  almost everywhere, we may apply

the dominated convergence theorem in order to obtain the third equality of the following computation:

$$\begin{aligned} \text{AVaR}_\alpha(X) &= \lim_{n \rightarrow \infty} \text{AVaR}_\alpha(X_n) = \lim_{n \rightarrow \infty} -\frac{1}{\alpha} \int_0^\alpha q_{X_n}(s) ds \\ &= -\frac{1}{\alpha} \int_0^\alpha q_X(s) ds, \end{aligned}$$

which proves (4.4).

The following example is the extension of the entropic risk measure to  $L^p$ . In case  $p \in [1, \infty)$ , we will see that the entropic risk measure is l.s.c. but not continuous.

**Example 4.2.** The entropic risk measure of parameter  $\lambda > 0$  is

$$\text{Entr}_\lambda(X) = \frac{1}{\lambda} \log E[e^{-\lambda X}], \quad X \in L^p.$$

$\text{Entr}_\lambda$  is a law-invariant convex risk measure on  $L^p$ . We denote its acceptance set by  $\mathcal{A}_\lambda$ . It is easily verified that  $\text{Entr}_\lambda$  is l.s.c.: suppose the sequence  $(X_n)_{n \in \mathbb{N}} \subset \mathcal{A}_\lambda$  converges to an  $X \in L^p$ . Then there is a subsequence  $(X_{n_i})_{i \in \mathbb{N}}$  which converges  $\mathbb{P}$ -a.s. to  $X$ , and by dominated convergence

$$\forall N \in \mathbb{N} : \quad E[e^{-\lambda X} \wedge N] = \lim_{i \rightarrow \infty} E[e^{-\lambda X_{n_i}} \wedge N] \leq 1.$$

Thus, by monotone convergence we have  $E[e^{-\lambda X}] \leq 1$ , implying that  $X \in \mathcal{A}_\lambda$ . Therefore,  $\mathcal{A}_\lambda$  is closed.

If  $p \in [1, \infty)$ , then  $\{X \in L^p \mid \text{Entr}_\lambda(X) = \infty\} \neq \emptyset$ , whereas, if  $p = \infty$ , then  $\text{dom } \text{Entr}_\lambda = L^\infty$ . Hence, by lemma 2.6,  $\text{Entr}_\lambda$  is continuous if and only if  $p = \infty$ .

Another common class of law-invariant convex risk measures on  $L^p$  are the semi-deviation risk measures  $\text{Dev}_r(\cdot) := -E[\cdot] + \|(\cdot - E[\cdot])^-\|_r$ . We will show that these coherent risk measures are always l.s.c., and they are continuous if and only if  $r \leq p$ .

**Example 4.3.** For any  $r \in [1, \infty]$

$$\text{Dev}_r(X) := -E[X] + \|(X - E[X])^-\|_r, \quad X \in L^p,$$

is a coherent risk measure on  $L^p$ . If  $1 \leq r \leq p$ , the Hölder inequality implies  $\text{Dev}_r \leq \text{Dev}_p$ . The Hölder inequality also implies that  $\text{Dev}_r$  is  $\|\cdot\|_p$ -continuous whenever  $r \leq p$ . Whereas, if  $r > p$ , then  $\{X \in L^p \mid \text{Dev}_r = \infty\} \neq \emptyset$ , so  $\text{Dev}_r$  cannot be continuous. Nevertheless, we will show that it is l.s.c. To this end, note that  $\{X \in L^p \mid \|X\|_r \leq k\}$ ,  $k \in \mathbb{R}$ , is closed in  $L^p$  (this is obvious for  $r = \infty$ , and due to the Fatou lemma else). Hence, the continuity of the mapping  $X \mapsto (X - E[X])^-$  implies that the sets  $\{X \in L^p \mid \|(X - E[X])^-\|_r \leq k\}$  are closed. Let  $(Y_n)_{n \in \mathbb{N}} \subset \mathcal{A}_{\text{Dev}_r}$  be a sequence converging to some  $Y \in L^p$ .

Then for every  $\epsilon > 0$  there exists an  $N(\epsilon) \in \mathbb{N}$  such that for all  $n \geq N(\epsilon)$ :  $E[Y_n] \leq E[Y] + \epsilon$ . Since  $Y_n \in \mathcal{A}_{\text{Dev}_r}$ , i.e.  $\|(Y_n - E[Y_n])^-\|_r \leq E[Y_n]$ , we have  $Y_n \in \{X \in L^p \mid \|(X - E[X])^-\|_r \leq E[Y] + \epsilon\}$  for all  $n \geq N(\epsilon)$ . The closedness of this latter set ensures that  $-E[Y] + \|(Y - E[Y])^-\|_r \leq \epsilon$ , and the arbitrariness of  $\epsilon > 0$  finally implies  $-E[Y] + \|(Y - E[Y])^-\|_r \leq 0$  or equivalently  $Y \in \mathcal{A}_{\text{Dev}_r}$ . Consequently, the acceptance set of  $\text{Dev}_r$  is closed in  $L^p$ , so  $\text{Dev}_r$  is l.s.c.

Clearly, as on  $L^\infty$ , the worst case risk measure is the most conservative convex risk measure on any  $L^p$ :

**Example 4.4.**  $-\text{essinf}(X)$ ,  $X \in L^p$ , is a l.s.c. law-invariant coherent risk measure on  $L^p$ . By cash-invariance and monotonicity, any convex risk measure  $\rho$  on  $L^p$  satisfies  $\rho(X) \leq \rho(0) - \text{essinf}(X)$ . Obviously, if  $p \in [1, \infty)$ , then  $\{X \in L^p \mid \text{essinf}(X) = -\infty\} \neq \emptyset$ . Hence,  $-\text{essinf}(\cdot)$  is only continuous if  $p = \infty$  (lemma 2.6).

The following example shows that in case  $p \in [1, \infty)$ , there are convex risk measures on  $L^p$  which might occur quite naturally, but which are not l.s.c. For instance, it seems natural to consider unbounded losses as unacceptable. Constructing an accordant risk measure, we could for instance start out by choosing any "nice" risk measure like AVaR or simply the expectation, and then knock out all positions having unbounded losses:

**Example 4.5.** For  $p \in [1, \infty)$ , consider

$$\rho : L^p \rightarrow (-\infty, \infty], \quad \rho(X) = -E[X] + \delta(X^- \mid L^\infty).$$

This law-invariant coherent risk measure assigns to an endowment  $X$  the value  $\infty$  in case the possible losses are not bounded, and  $E[-X]$  else. Clearly, the acceptance set  $\mathcal{A}_\rho$  is not closed, so  $\rho$  is not l.s.c. For  $Z \in \mathcal{P}_p$  we have that

$$\begin{aligned} \rho^*(Z) &= \sup_{X \in L^p} E[XZ] - \rho(X) = \sup_{X \in \{Y \in L^p \mid Y^- \in L^\infty\}} E[X(Z+1)] \\ &\geq \sup_{k \in \mathbb{R}} E[k(Z+1)1_{\{Z > -1\}}]. \end{aligned}$$

Hence,  $\text{dom } \rho^* = \{-1\}$  and  $\rho^* = \delta(\cdot \mid \{-1\}) = -E^*$ , and thus  $\text{cl}(\rho) = -E$ .

In fact, not every l.s.c. convex risk measure on  $L^r$ , for some  $r > p$ , can be extended to  $L^p$ . For instance choose  $Z \in (L^r)^* \setminus (L^p)^*$  and consider  $X \mapsto E[XZ]$ . Clearly, this continuous coherent risk measure on  $L^r$  fails to be proper on  $L^p$ . The following example presents a coherent risk measure on  $L^\infty$  which cannot be extended to  $L^p$ , for any  $p \in [1, \infty)$ . In particular, this example illustrates theorem 3.1.

**Example 4.6.** Let  $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1], \mathcal{B}(0, 1], \lambda)$  where  $\lambda$  denotes the Lebesgue measure restricted to the Borel- $\sigma$ -algebra  $\mathcal{B}(0, 1]$ , and let  $A_n := (0, \frac{1}{2^n}]$ ,  $n \in \mathbb{N}$ . Moreover, let  $\mathbb{P}_n(\cdot) := \mathbb{P}(\cdot \mid A_n)$ , and we denote by  $\text{essinf}_{\mathbb{P}_n}(X)$  the essential infimum of a random variable  $X$  under the measure  $\mathbb{P}_n$ . Define

$$\rho(X) := \lim_{n \rightarrow \infty} -\text{essinf}_{\mathbb{P}_n}(X), \quad X \in L^p.$$

The function  $\rho$  is convex, monotone, and cash-invariant. But if  $p \in [1, \infty)$ ,  $\rho$  fails to be proper because  $\{X \in L^p \mid \rho(X) = -\infty\} \neq \emptyset$ . Moreover, note that  $\{X \in L^p \mid \rho(X) \leq 0\} = L^p$ .

However, if  $p = \infty$ , then  $\rho$  is a coherent risk measure. The domain of its conjugate function is concentrated on  $ba \setminus L^1$ , because for any  $Z \in \mathcal{P}_\infty \cap L^1$ :

$$\rho^*(Z) = \sup_{X \in \mathcal{A}_\rho} E[XZ] \geq \sup_{k, n \in \mathbb{N}} E[-k1_{A_n^c} Z] = \sup_{k, n \in \mathbb{N}} k(1 + E[Z1_{A_n}]) = \infty.$$

That is, condition (iii) of theorem 3.1 is not satisfied. Hence,  $\rho$  is a coherent risk measure on  $L^\infty$ , which cannot be extended to  $L^p$ , for any  $p \in [1, \infty)$ .

The extension and restriction operations as outlined in section 3 do not commute in general. The following example illustrates this, thereby illuminating theorem 3.2.

**Example 4.7.** In this example we will construct a l.s.c. coherent risk measure on  $L^1$  via a suitable acceptance set  $\mathcal{A}$  such that  $\mathcal{A} \cap L^\infty$  is not  $L^1$ -dense in  $\mathcal{A}$ . Let  $T$  be a shifted exponentially distributed random variable with density

$$f(x) = \exp(-(x+2))1_{\{x \geq -2\}}(x).$$

Furthermore, let  $B$  be the monotone hull of the convex hull  $\text{conv}(T, L_+^1)$  of  $T$  and  $L_+^1$ , i.e.

$$B = \{X \in L^1 \mid \exists Y \in \text{conv}(T, L_+^1) : X \geq Y\}.$$

Finally, let

$$A := \{tX \mid X \in B, t \geq 0\}$$

and  $\mathcal{A} := \overline{A}$ . Then  $\mathcal{A}$  is closed, convex, and satisfies the conditions (2.1) for  $p = 1$ . Hence, by lemmas 2.3 and 2.4,  $\rho_{\mathcal{A}}$  is a l.s.c. convex risk measure on  $L^1$  with acceptance set  $\mathcal{A}_\rho = \mathcal{A}$ . Moreover, since  $\mathcal{A}$  is a cone,  $\rho$  is also coherent.

We claim that

$$\mathcal{A}_{\rho_\infty} = \mathcal{A} \cap L^\infty = L_+^\infty, \tag{4.5}$$

implying that  $\rho_\infty = -\text{essinf}$ . In particular,  $\mathcal{A} \cap L^\infty$  is not  $L^1$ -dense in  $\mathcal{A}$  and  $(\rho_\infty)^1 \neq \rho$ , see conditions (iv) and (v) in theorem 3.2.

As for the proof of (4.5), the inclusion  $L_+^\infty \subset \mathcal{A} \cap L^\infty$  follows by construction. To show the converse,  $L^\infty \setminus L_+^\infty \subset L^\infty \setminus (\mathcal{A} \cap L^\infty)$ , we choose any  $S \in L^\infty$  such that  $\mathbb{P}(S < 0) > 0$ . Since  $S \notin L_+^1$  is bounded whereas  $T$  is unbounded from above,  $S$  cannot be an element of the convex hull  $\text{conv}(T, L_+^1)$ , and neither of its monotone hull  $B$  because any convex combination in  $\text{conv}(T, L_+^1)$  is either  $\mathbb{P}$ -a.s. positive or unbounded from above, so it cannot be dominated by  $S$ . But then, clearly  $S \notin A$  too. Now suppose  $S \in \mathcal{A} \setminus A$ . Then there would be a sequence  $(S_n)_{n \in \mathbb{N}} \subset A$  converging to  $S$  in  $L^1$ . By monotonicity of  $A$  we may assume that  $S_n \geq S$  for all  $n \in \mathbb{N}$  (otherwise  $\tilde{S}_n := S_n \vee S \in A$  will do), and shifting to a subsequence if necessary, we may assume that  $S_n \rightarrow S$   $\mathbb{P}$ -a.s. Hence there is some  $N_0 \in \mathbb{N}$  such that  $\mathbb{P}(S_n < 0) > 0$  for all  $n \geq N_0$ . By construction of  $A$

there are  $t_n > 0$ ,  $\alpha_n \in (0, 1]$  and  $X_n \in L_+^1$  such that  $S_n \geq t_n(\alpha_n T + (1 - \alpha_n)X_n)$ ,  $n \geq N_0$ . Thus  $\{S_n < 0\} \subset \{T < 0\}$ , and consequently we have  $\mathbb{P}$ -a.s. that

$$\{S < 0\} = \bigcup_{k \in \mathbb{N}} \bigcap_{m \geq k} \{S_m < 0\} \subset \{T < 0\}.$$

Let  $C := \{S < 0\}$ . Then  $\mathbb{P}(C) > 0$  and  $\epsilon := E[-S1_C] > 0$  and  $\delta := E[-T1_C] > 0$ . Choose  $N_1 \geq N_0$  such that for all  $n \geq N_1$ :  $\|S - S_n\|_1 \leq \frac{\epsilon}{2}$ . For  $n \geq N_1$  we have that:

$$\begin{aligned} \frac{\epsilon}{2} \geq E[S_n - S] &\geq E[(S_n - S)1_C] \geq E[(t_n \alpha_n T + t_n(1 - \alpha_n)X_n - S)1_C] \\ &\geq t_n \alpha_n E[T1_C] + E[-S1_C] = -t_n \alpha_n \delta + \epsilon. \end{aligned}$$

Consequently,  $t_n \alpha_n \geq \frac{\epsilon}{2\delta} =: r > 0$ , and thus  $S_n \geq rT$ , for all  $n \geq N_1$ . But this contradicts the boundedness of  $S$ . Hence  $S \notin \mathcal{A}$  and (4.5) is proved.

We close this section by giving yet another example of a non-l.s.c. convex risk measure. This example illustrates problems which are particularly related to section B, where we study law-invariant convex functions on  $L^p$ .

**Example 4.8.** Let  $A := \{X \in L^1 \mid X \geq -2, \forall r \geq 0 : \mathbb{P}(X > r) > 0\}$  and  $\mathcal{A} := L_+^1 \cup A$ .  $\mathcal{A}$  is convex and law-invariant, and satisfies the conditions (2.1). By lemma 2.4,  $\rho_{\mathcal{A}}$  is a convex risk measure on  $L^1$ , and it is easily verified that  $\mathcal{A}_{\rho} = \mathcal{A}$ , so  $\rho_{\mathcal{A}}$  is law-invariant. Let  $T$  be the random variable from example 4.7. Clearly,  $T \in \mathcal{A}$ . Observe that  $\mathcal{A}$  cannot be closed because on the one hand  $-1 \notin \mathcal{A}$ , whereas on the other hand lemmas 2.4 and B.8 imply that  $\text{cl}(\rho)(-1) \leq 0$ , i.e.  $-1 = E[T] \in \overline{\mathcal{A}}$ . In other words, shifting from  $\rho$  to its closure, the almost sure loss of 1 is suddenly acceptable. In particular, we note that  $\rho_{\mathcal{A}}$  is not  $\succeq$ -monotone (see definition B.4).

## A Some Facts from Convex Analysis

For the convenience of the reader we collect here some standard definitions and results in convex analysis. For more background we refer to Rockafellar [14] and Ekeland and Témam [6].

Let  $E$  denote a Hausdorff locally convex topological vector space with topological dual  $E^*$ . A function  $f : E \rightarrow [-\infty, +\infty]$  is *convex* if

$$f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y) \quad \forall X, Y \in E, \quad \forall \lambda \in [0, 1],$$

whenever the right-hand side is defined. We write  $\text{dom } f = \{f < \infty\}$  for the (effective) domain of  $f$ . We call  $f$  *proper* if  $f > -\infty$  and  $\text{dom } f \neq \emptyset$ .

$f$  is said to be *lower semi-continuous* (l.s.c.) if the level sets  $\{X \in E \mid f(X) \leq k\}$  are closed for all  $k \in \mathbb{R}$ , or equivalently, if for any net  $(X_\alpha)_{\alpha \in D} \subset E$  converging to some  $X \in E$  we have that  $f(X) \leq \liminf_{\alpha} f(X_\alpha)$ . This property is also equivalent to  $\text{epi } f = \{(X, a) \in E \times \mathbb{R} \mid f(X) \leq a\}$  being a closed set in  $E \times \mathbb{R}$  equipped with the product topology (see e.g. [6] proposition 2.3).

A convex set  $\mathcal{C} \subset E$  is closed if and only if it is  $\sigma(E, E^*)$ -closed. As a consequence, a convex function  $f$  is l.s.c. if and only if  $f$  is l.s.c. with respect to  $\sigma(E, E^*)$ .

The *closure* of  $f$  is denoted by  $\text{cl}(f)$  and defined as  $\text{cl}(f) \equiv -\infty$ , if  $f(X) = -\infty$  for some  $X$ , and as greatest convex l.s.c. function majorised by  $f$ , else. Note that a l.s.c. convex function which assumes the value  $-\infty$  cannot take any finite value. Hence, if  $f$  is proper l.s.c. and convex, then  $\text{cl}(f) = f$ .

The *conjugate function* of a function  $f : E \rightarrow [-\infty, +\infty]$ ,

$$f^*(\mu) = \sup_{X \in E} (\langle \mu, X \rangle - f(X)),$$

is a l.s.c. convex function on  $E^*$ . Moreover,  $(\text{cl}(f))^* = f^*$ , and the following convex duality relation holds (proposition 4.1 in [6])

$$f^{**} = \text{cl}(f). \quad (\text{A.6})$$

As for the subgradients, we have (proposition 5.1 in [6])

$$\mu \in \partial f(X) \Leftrightarrow f(X) + f^*(\mu) = \langle \mu, X \rangle. \quad (\text{A.7})$$

The *indicator function* of a set  $\mathcal{C} \subset E$  is defined as

$$\delta(X | \mathcal{C}) := \begin{cases} 0, & X \in \mathcal{C} \\ +\infty, & X \notin \mathcal{C}. \end{cases}$$

$\delta(\cdot | \mathcal{C})$  is convex and l.s.c. if and only if  $\mathcal{C}$  is convex and closed. Its conjugate is the *support function* of  $\mathcal{C}$ ,

$$\delta^*(\mu | \mathcal{C}) = \sup_{X \in \mathcal{C}} \langle \mu, X \rangle.$$

Notice that  $E$  and  $E^*$  can be interchanged in the definition of  $\delta$  and  $\delta^*$ .

## B Law Invariant Convex Functions on $L^p$

One of the main ingredients for proving theorem 3.4 is lemma B.2. In fact this lemma is an extension of some results by Jouini, Schachermayer, and Touzi in [12]. In this context we will have to do some topological studies on  $L^p$ .

Let  $\mu \in ba$  and  $\mathcal{G} \subset \mathcal{F}$  be a sub- $\sigma$ -algebra. As in [12] we define the conditional expectation  $E[\mu | \mathcal{G}]$  of  $\mu$  by

$$E[\cdot | \mathcal{G}] : ba \rightarrow ba, \quad \langle E[\mu | \mathcal{G}], X \rangle := \langle \mu, E[X | \mathcal{G}] \rangle \quad \forall X \in L^\infty.$$

This definition is consistent with the ordinary conditional expectation in case  $\mu \in L^1 \subset ba$ .

**Remark B.1.** Let  $\mathcal{G} = \sigma(A_1, \dots, A_n)$  be finite. Then, for all  $X \in L^\infty$ ,

$$\langle E[\mu | \mathcal{G}], X \rangle = \langle \mu, E[X | \mathcal{G}] \rangle = \sum_{i=1}^n E[X 1_{A_i}] \frac{\mu(A_i)}{\mathbb{P}(A_i)} = E[ZX]$$

for  $Z = \sum_{i=1}^n \frac{\mu(A_i)}{\mathbb{P}(A_i)} 1_{A_i} \in L^\infty$ .

Note that  $(L^p, L^r)$  is a dual pair for every  $r \in [\frac{p}{p-1}, \infty]$ , where we set  $\frac{1}{0} = \infty$  and  $\frac{\infty}{\infty-1} = 1$ .

**Lemma B.2.** *Let  $f : L^p \rightarrow (-\infty, \infty]$  be a proper l.s.c. law-invariant convex function. Then  $f$  is also l.s.c. with respect to any  $\sigma(L^p, L^r)$ -topology for every  $r \in [\frac{p}{p-1}, \infty]$ .*

When proving this, we will apply the following lemma:

**Lemma B.3.** *Let  $D \subset L^p$  be a  $\|\cdot\|_p$ -closed convex law-invariant set. Then  $D$  is  $\sigma(L^p, L^r)$ -closed for every  $r \in [\frac{p}{p-1}, \infty]$ .*

*Proof.* 0. If  $D = \emptyset$ , the assertion is obvious. For the remainder of this proof, we assume that  $D \neq \emptyset$ .

1. According to lemma 4.2 in [12], for all  $Y \in D$  and all sub- $\sigma$ -algebras  $\mathcal{G} \subset \mathcal{F}$  we have that  $E[Y | \mathcal{G}] \in D$ .

2. Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence in  $D$  converging to some  $X \in L^p$  in the  $\sigma(L^p, L^r)$ -topological sense, i.e.  $\lim_{n \rightarrow \infty} E[X_n Z] = E[X Z]$  for all  $Z \in L^r$ . Then, in view of remark B.1, if  $\mathcal{G}$  is finite, we have  $\langle E[\mu | \mathcal{G}], X_n \rangle \rightarrow \langle E[\mu | \mathcal{G}], X \rangle$  for all  $\mu \in (L^p)^*$ . But by definition, this equals  $\langle \mu, E[X_n | \mathcal{G}] \rangle \rightarrow \langle \mu, E[X | \mathcal{G}] \rangle$  for all  $\mu \in (L^p)^*$ . Since  $E[X_n | \mathcal{G}] \in D$  for all  $n \in \mathbb{N}$  according to 1., we conclude that  $E[X | \mathcal{G}] \in D$  because  $D$  is closed and convex and thus  $\sigma(L^p, (L^p)^*)$ -closed. Hence,  $E[X | \mathcal{G}] \in D$  for all finite sub- $\sigma$ -algebras  $\mathcal{G} \subset \mathcal{F}$ . Recalling that we can approximate  $X$  in  $(L^p, \|\cdot\|_p)$  by a sequence of conditional expectations  $(E[X | \mathcal{G}_i])_{i \in \mathbb{N}}$  in which the  $\mathcal{G}_i$ s are all finite, we conclude by means of the norm-closedness of  $D$  that  $X \in D$ . Thus  $D$  is  $\sigma(L^p, L^r)$ -closed.  $\square$

*Proof of lemma B.2.* For every  $k \in \mathbb{R}$  the level sets  $\{X \in L^p \mid f(X) \leq k\}$  are  $\|\cdot\|_p$ -closed, convex, and law-invariant. Hence, lemma B.3 yields the  $\sigma(L^p, L^r)$ -closedness of the level sets, i.e.  $f$  is l.s.c. with respect to the  $\sigma(L^p, L^r)$ -topology.  $\square$

Besides lemma B.2, lemmas B.6 and B.8 below form the basis for the proof of theorem 3.4. It will require some working with quantiles and some further analysis of law-invariant convex functions in order to deriving these lemmas.

Recall that the (left continuous) quantile function of a random variable  $X$  is

$$q_X : (0, 1) \rightarrow \mathbb{R}, \quad q_X(s) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(X \leq x) \geq s\}. \quad (\text{B.8})$$

The following properties of quantiles will be frequently used in our proofs: For all  $a \in \mathbb{R}$  and  $t > 0$  we have  $q_{tX+a} = tq_X + a$ . If  $Y$  is another random variable

such that  $X \leq Y$ , then  $q_X \leq q_Y$ . Moreover,  $q_{X^+} = (q_X)^+$  and  $q_{-X^-} = -(q_X)^-$ , and thus

$$|q_X| = q_{X^+} - q_{-X^-}. \quad (\text{B.9})$$

We will now introduce two orders on  $L^1$  which are well-known from utility theory. To this end, recall that a utility function is a strictly concave and strictly increasing function  $u : \mathbb{R} \rightarrow \mathbb{R}$ .

**Definition B.4.** For any two  $X, Y \in L^1$  we define

(i) the concave order:

$$X \succeq_c Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \text{ for all concave functions } u : \mathbb{R} \rightarrow \mathbb{R},$$

(ii) the second order stochastic order:

$$X \succeq Y \Leftrightarrow E[u(X)] \geq E[u(Y)] \text{ for all utility functions } u.$$

A function  $f : L^p \rightarrow [-\infty, \infty]$  is said to be  $\succeq_c$  ( $\succeq$ )-monotone if  $f(X) \leq f(Y)$  whenever  $X \succeq_c Y$  ( $X \succeq Y$ ).

Clearly, both  $X \geq Y$  and  $X \succeq_c Y$  imply  $X \succeq Y$ . However,  $\geq$  and  $\succeq_c$  are not related in general.

In the sequel we will build on the following three facts: for  $X, Y \in L^1$  we have

$$X \succeq Y \Leftrightarrow \int_0^t q_X(s) ds \geq \int_0^t q_Y(s) ds \text{ for all } 0 < t \leq 1, \quad (\text{B.10})$$

and

$$X \succeq_c Y \Leftrightarrow X \succeq Y \text{ and } E[X] = E[Y]. \quad (\text{B.11})$$

Moreover, if  $X \in L^p$ ,  $p \in [1, \infty)$ , and  $Y \in L^q$  where  $q := \frac{p}{p-1}$ , then the Hardy-Littlewood inequalities hold, i.e.

$$\int_0^1 q_X(1-s)q_Y(s) ds \leq E[XY] \leq \int_0^1 q_X(s)q_Y(s) ds. \quad (\text{B.12})$$

For a proof of (B.10), (B.11) and (B.12), we refer to theorems 2.58, A.24, and corollary 2.62 in [10].

**Lemma B.5.** Let  $p \in [1, \infty)$  and  $q := \frac{p}{p-1}$ . For  $X \in L^p$  and  $Z \in L^q$  we have that

$$\int_0^1 q_X(s)q_Z(s) ds = \sup_{\tilde{X} \sim X} E[\tilde{X}Z] = \sup_{\tilde{Z} \sim Z} E[X\tilde{Z}].$$

Lemma B.5 is partly stated and proved by Föllmer/Schied in [10] lemma 4.55 for the case  $X \in L^1$  and  $Z \in L^\infty$ . Their result in turn can be extended to  $X \in L^p$  and  $Z \in L^q$  by suitable approximation. For the sake of completeness, we provide a self-contained proof:

*Proof.* By (B.12) we always have that

$$\sup_{\tilde{X} \sim X} E[\tilde{X}Z] \leq \int_0^1 q_X(s)q_Z(s) ds.$$

Suppose the distribution function  $F_Z$  of  $Z$  is continuous. Then  $U := F_Z(Z)$  has a uniform distribution on  $(0, 1)$  and  $Z = q_Z(U)$   $\mathbb{P}$ -a.s.. Clearly, for  $\tilde{X} := q_X(U) \sim X$  we have

$$E[\tilde{X}Z] = E[q_X(U)q_Z(U)] = \int_0^1 q_X(s)q_Z(s) ds.$$

Now suppose  $Z$  has no continuous distribution. Denote by  $D$  the countable set of all  $z \in \mathbb{R}$  such that  $\mathbb{P}[Z = z] > 0$ . W.l.o.g. (by adding a constant to  $Z$  if necessary) we will assume that  $0 \notin D$ . Let  $A_z := \{Z = z\}$ ,  $z \in D$ . Since  $(\Omega, \mathcal{F}, \mathbb{P})$  atom-less, for each  $z \in D$  there is a random variable  $U_z$  being uniformly distributed on  $(0, \frac{|z|}{2})$  under the measure  $\mathbb{P}(\cdot | A_z)$ . We claim that the law of

$$Z_n := Z - \frac{1}{n} \sum_{z \in D} \text{sgn}(z)U_z 1_{A_z}$$

is continuous. Indeed, for any  $y \in \mathbb{R}$

$$\begin{aligned} \mathbb{P}(Z_n = y) &= \mathbb{P}(Z_n = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(Z = z, U_z = \text{sgn}(z)n(z - y)) \\ &= \mathbb{P}(Z = y, Z \notin D) + \sum_{z \in D} \mathbb{P}(A_z)\mathbb{P}(U_z = \text{sgn}(z)n(z - y) | A_z) \\ &= 0. \end{aligned}$$

Observe that  $Z_n^+ \leq Z^+$ ,  $Z_n^- \leq Z^-$  for all  $n \in \mathbb{N}$ , and thus  $Z_n \in L^q$  and  $|q_{Z_n}| \leq |q_Z|$  by (B.9). Moreover,  $Z_n$  converges to  $Z$   $\mathbb{P}$ -a.s. and in  $L^q$ . In particular, the respective quantile functions converge almost everywhere. Therefore, the sequence  $(q_X q_{Z_n})_{n \in \mathbb{N}} \subset L^1(0, 1)$  converges almost everywhere to  $q_X q_Z \in L^1(0, 1)$ , and we have  $|q_X q_{Z_n}| \leq |q_X q_Z|$ . Hence, the dominated convergence theorem in combination with our result from the beginning of the proof yields

$$\begin{aligned} \int_0^1 q_X(s)q_Z(s) ds &= \lim_{n \rightarrow \infty} \int_0^1 q_X(s)q_{Z_n}(s) ds \\ &= \lim_{n \rightarrow \infty} \sup_{\tilde{X} \sim X} E[\tilde{X}Z_n] = \sup_{\tilde{X} \sim X} E[\tilde{X}Z] \end{aligned}$$

where the last equality follows from

$$|E[\tilde{X}Z_n] - E[\tilde{X}Z]| \leq \|\tilde{X}\|_p \|Z_n - Z\|_q \leq \frac{1}{n} \|X\|_p \|Z\|_q \quad \text{for all } \tilde{X} \sim X.$$

Finally, reversing the roles of  $X$  and  $Z$  completes the proof.  $\square$

**Lemma B.6.** (compare to theorem 4.54 in [10]) Let  $f : L^p \rightarrow (-\infty, \infty]$  be a proper l.s.c. convex function on  $L^p$ . Then the following are equivalent:

- (i)  $f$  is law-invariant.
- (ii)  $f^*$  (resp.  $f^*|_{L^1}$  if  $p = \infty$ ) is law-invariant.

Moreover, if either holds, then:

$$f^*(Z) = \sup_{X \in L^p} \int_0^1 q_X(s)q_Z(s) ds - f(X), \quad Z \in (L^p)^* \cap L^1,$$

and

$$f(X) = \sup_{Z \in (L^p)^* \cap L^1} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z), \quad X \in L^p.$$

*Proof.* (i)  $\Rightarrow$  (ii): For any  $Z \in (L^p)^* \cap L^1$  lemma B.5 yields

$$\begin{aligned} f^*(Z) &= \sup_{X \in L^p} E[XZ] - f(X) = \sup_{X \in L^p} \left( \sup_{\tilde{X} \sim X} E[\tilde{X}Z] \right) - f(X) \\ &= \sup_{X \in L^p} \int_0^1 q_X(s)q_Z(s) ds - f(X) \end{aligned}$$

in which the latter expression depends on the law of  $Z$  only.

(ii)  $\Rightarrow$  (i): In view of lemma B.2 for the case  $p = \infty$ , and by lemma B.5:

$$\begin{aligned} f(X) &= f^{**}(X) = \sup_{Z \in (L^p)^* \cap L^1} \left( \sup_{\tilde{Z} \sim Z} E[X\tilde{Z}] \right) - f^*(Z) \\ &= \sup_{Z \in (L^p)^* \cap L^1} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z) \end{aligned}$$

for all  $X \in L^p$ . Hence,  $f$  is law-invariant.

Finally, the stated representations are clear by now.  $\square$

The proof of lemma B.8 draws heavily on the following result, which can essentially be found in Dana [4]. For convenience of the reader, we give a self-contained proof below.

**Lemma B.7.** For  $X, Y \in L^1$ :

- (i)  $X \succeq Y$  if and only if

$$\int_0^1 q_X(s)f(s) ds \leq \int_0^1 q_Y(s)f(s) ds$$

for all increasing  $f : (0, 1) \rightarrow (-\infty, 0]$  such that both integrals exist.

(ii)  $X \succeq_c Y$  if and only if

$$\int_0^1 q_X(s)f(s) ds \leq \int_0^1 q_Y(s)f(s) ds$$

for all increasing  $f : (0, 1) \rightarrow \mathbb{R}$  such that both integrals exist.

*Proof.* (i): " $\Leftarrow$ ": Note that the functions  $-1_{(0,t)}(\cdot)$  are increasing for all  $0 < t \leq 1$ . Now apply (B.10).

" $\Rightarrow$ ": Let  $X \succeq Y$ .

1. In a first step we assume that  $f$  is a simple function, i.e.

$$f(s) = \sum_{i=1}^{n-1} a_i 1_{(t_{i-1}, t_i]}(s) + a_n 1_{(t_{n-1}, t_n)}(s)$$

where  $t_0 = 0 < t_1 < \dots < t_n = 1$  is a finite partition of  $(0, 1)$ , and  $a_i \in \mathbb{R}$  such that  $a_1 \leq a_2 \leq \dots \leq a_n \leq 0$ . Applying (B.10) we have

$$a_n \int_0^1 q_X(s) ds \leq a_n \int_0^1 q_Y(s) ds,$$

and for  $j = 1, \dots, n-1$ :

$$(a_j - a_{j+1}) \int_0^{t_j} q_X(s) ds \leq (a_j - a_{j+1}) \int_0^{t_j} q_Y(s) ds.$$

Summing up these inequalities we arrive at

$$\sum_{j=1}^n a_j \int_{t_{j-1}}^{t_j} q_X(s) ds \leq \sum_{j=1}^n a_j \int_{t_{j-1}}^{t_j} q_Y(s) ds$$

and thus the desired

$$\int_0^1 q_X(s)f(s) ds \leq \int_0^1 q_Y(s)f(s) ds.$$

2. Now for general  $f$ , we approximate  $f$  by simple functions

$$f_k(s) := \begin{cases} -\frac{i-1}{2^k} & \text{if } f(s) \in (-\frac{i}{2^k}, -\frac{i-1}{2^k}] \text{ for a } i = 1, \dots, k2^k \\ -k & \text{else} \end{cases}, \quad k \in \mathbb{N}.$$

The functions  $f_k$  converge to  $f$  pointwise and  $|f_k| \leq |f|$  for all  $k \in \mathbb{N}$ . Hence, the dominated convergence theorem in conjunction with 1. yields

$$\begin{aligned} \int_0^1 q_Y(s)f(s) ds &= \lim_{k \rightarrow \infty} \int_0^1 q_Y(s)f_k(s) ds \geq \lim_{k \rightarrow \infty} \int_0^1 q_X(s)f_k(s) ds \\ &= \int_0^1 q_X(s)f(s) ds. \end{aligned}$$

(ii): " $\Rightarrow$ ": Let  $X \succeq_c Y$ . By (B.11) we know that  $X \succeq Y$  and  $\int_0^1 q_X(s) ds = \int_0^1 q_Y(s) ds$ . Hence, by (i), adding up inequalities, we deduce that

$$\int_0^1 q_X(s)f(s) ds \leq \int_0^1 q_Y(s)f(s) ds$$

for all increasing  $f : (0, 1) \rightarrow \mathbb{R}$  which are bounded from above. Finally, the usual monotone approximation argument from integration theory yields the assertion for any increasing  $f : (0, 1) \rightarrow \mathbb{R}$ .

" $\Leftarrow$ ": Again, simply apply (B.11) and (i).  $\square$

The following lemma is a main result in Dana [4] where she proves it for  $p \in \{1, \infty\}$ . In the proof of theorem 3.4 we need this result for general  $p \in [1, \infty]$ , so we provide a self-contained proof below.

**Lemma B.8.** *Let  $f : L^p \rightarrow (-\infty, \infty]$  be a proper l.s.c. convex function. Equivalent are:*

- (i)  $f$  is law-invariant.
- (ii)  $f$  is  $\succeq_c$ -monotone.

Moreover, if in addition  $f$  is monotone, then (i) is equivalent to

- (iii)  $f$  is  $\succeq$ -monotone.

In particular, if either of the conditions (i), (ii) or (iii) holds, then

- (iv)  $f(E[X | \mathcal{G}]) \leq f(X)$  for all  $X \in L^p$  and all sub- $\sigma$ -algebras  $\mathcal{G} \subset \mathcal{F}$ .

*Proof.* Let  $q = \frac{p}{p-1}$  if  $p \in [1, \infty)$ , and, in view of lemma B.2,  $q = 1$  if  $p = \infty$ .

(i)  $\Rightarrow$  (ii): Let  $X \succeq_c Y$ . Then by lemma B.6 and lemma B.7 (ii):

$$\begin{aligned} f(X) &= \sup_{Z \in L^q} \int_0^\infty q_X(s)q_Z(s) ds - f^*(Z) \\ &\leq \sup_{Z \in L^q} \int_0^\infty q_Y(s)q_Z(s) ds - f^*(Z) = f(Y). \end{aligned}$$

(ii)  $\Rightarrow$  (i): Conversely, suppose that  $f$  is  $\succeq_c$ -monotone and let  $X \sim Y$ . Trivially,  $X \succeq_c Y$  and  $Y \succeq_c X$ , so  $f(X) = f(Y)$ .

(i)  $\Rightarrow$  (iii): Let  $X \succeq Y$ . Since  $f$  is monotone, we have that  $\text{dom } f^* \subset L^q_-$  (see [7] lemma 3.2). Hence, by lemmas B.6 and B.7 (i),

$$\begin{aligned} f(X) &= \sup_{Z \in L^q_-} \int_0^1 q_X(s)q_Z(s) ds - f^*(Z) \\ &\leq \sup_{Z \in L^q_-} \int_0^1 q_Y(s)q_Z(s) ds - f^*(Z) = f(Y). \end{aligned}$$

- (iii)  $\Rightarrow$  (i): If  $X \sim Y$ , then  $X \succeq Y$  and  $Y \succeq X$ . Thus,  $f(X) = f(Y)$ .  
 (iv): For any concave function  $u : \mathbb{R} \rightarrow \mathbb{R}$  (so in particular for every utility function) the Jensen inequality yields

$$E[u(E[X | \mathcal{G}])] \geq E[E[u(X) | \mathcal{G}]] = E[u(X)].$$

Hence,  $E[X | \mathcal{G}] \succeq_c X$ . Now apply (ii) or (iii). □

On the one hand, the required l.s.c. in lemma B.8 cannot be dropped. This is for instance illustrated by example 4.8. On the other hand, example 4.5 shows that there are non-l.s.c. convex risk measures which are  $\succeq$ -monotone. Finally, note that the proof of lemma B.8 relies on lemma B.2, and thus on lemma 4.2 in [12], only in case  $p = \infty$ . We recall that lemma 4.2 in [12] states that if  $\emptyset \neq D \subset L^p$  is a convex law-invariant and  $\|\cdot\|_p$ -closed set, then  $E[X | \mathcal{G}] \in D$  for all  $X \in D$  and all sub- $\sigma$ -algebras  $\mathcal{G} \subset \mathcal{F}$ . In fact, for every such set  $D$ , the indicator function  $\delta(\cdot | D)$  is a proper l.s.c. law-invariant convex function. Therefore, by lemma B.8 (iv),  $\delta(E[X | \mathcal{G}] | D) \leq \delta(X | D)$  which implies  $E[X | \mathcal{G}] \in D$  if  $X \in D$ . Hence, we have derived an alternative proof of lemma 4.2 in [12] for the cases  $p \in [1, \infty)$  (but clearly not for  $p = \infty$ ).

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