Tutorial Sheet 3

Problem 1. Prove that $C^\infty_0(\mathbb{R}^d) \subseteq S(\mathbb{R}^d)$. (Hint: Example 1.3 on p. 3 in [LP].)

Problem 2. (a) Prove that for all $m \in \mathbb{N}_0$ there exists a $C \geq 1$ such that
\[
\frac{1}{C} |x|^m \leq \sum_{\substack{\gamma \in \mathbb{N}_0^d \\ |\gamma|=m}} |x^\gamma| \leq C |x|^m, \quad \forall x \in \mathbb{R}^d.
\]
(Hint: Consider the case of $x$ in the unit sphere $\partial B_1 \subset \mathbb{R}^d$ first.)
(b) Prove that for all $\varphi \in S(\mathbb{R}^d)$, $\alpha, \beta \in \mathbb{N}_0^d$ and $N \in \mathbb{N}$, there exists a $C > 0$ such that
\[
\left| x^\alpha \partial^\beta \varphi(x) \right| \leq \frac{C}{1 + |x|^N}, \quad \forall x \in \mathbb{R}^d.
\]
(c) Prove that $S(\mathbb{R}^d) \subseteq L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.
(Hence, we have in particular $S(\mathbb{R}^d) \subseteq L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.)

Problem 3. Prove that for any sequence $(\varphi_j)_{j \in \mathbb{N}}$ in $S(\mathbb{R}^d)$, with $\varphi_j \to 0$ in $S(\mathbb{R}^d)$ as $j \to \infty$, we have $\varphi_j \to 0$ in $L^p(\mathbb{R}^d)$ for all $p \in [1, \infty]$.
(Hence, the embedding $S(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is continuous for all $p \in [1, \infty]$.)

Problem 4. (a) Verify Example 1.8 on pp. 9-10 in [LP].
(b) Verify Example 1.9 on p. 10 in [LP].
(c) Verify Example 1.10 on pp. 10-11 in [LP].

This tutorial sheet is to be solved during the tutorial class on 15.05.2019.