Witten's Approach to Morse Theory

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July 23, 2020

Morse Theory

Morse theory deals with a compact smooth manifold M, and on it a smooth function $h : M \to \mathbb{R}$.

- The points $p \in M$ such that $\nabla h(p) = 0$ are called **critical points**.
- **Morse functions** are those h such that the **Hessian**

$$H_h(X, Y) = X \cdot (Y \cdot f)(p) - (\nabla_X Y) \cdot f(p)$$

  is non-singular for every $p \in \mathrm{Crit}(p)$.
- The **index** of a critical point p is the number of negative eigenvalues of its Hessian $H_h(p)$.
- (Morse Lemma) There are local coordinates (x_j) at each critical point p such that h is a diagonal quadratic form

$$h(p) = \frac{1}{2} \sum \lambda_j x_j^2$$

  with λ_j negative precisely for $j = 1, ..., \mathrm{index}(p)$.

Morse Inequalities

Theorem (Morse Inequalities)

Let h be a Morse function on the compact manifold M. Let β_j denote the j-th Betti number $b_j(M) = dim\, H^j_{dR}(M)$ and let ν_j denote the number of critical points of index j. Then we have the inequality

$$\sum_{j=1}^{k} (-1)^j \beta_j \leq \sum_{j=1}^{k} (-1)^j \nu_j$$

with equality when $k = dim\, M$.

- A standard proof could be found in Milnor, Morse Theory, Ch 5.
- In this talk, we prove this theorem via an alternative approach by means of the perturbed de Rham complex on M.

The Witten complex

Definition (Perturbed de Rham Operator)

Consider the Clifford bundle $S = \bigwedge^* T^*M$ on M. Let $s \in \mathbb{R}^+$. The **perturbed de Rham operator** is given by

$$D_s = d_s + d_s^* = D + sR$$

where R is a self-adjoint endomorphism of S given by $dh \wedge - dh \lrcorner$.

- In terms of the endomorphism $\mathbf{H}_h$ of $\bigwedge^* T^*M$ defined in terms of the Hessian relative to an orthonormal frame e_i,

$$\mathbf{H}_h = \sum_{i,j} H_h(e_i, e_j) L_{e_i} R_{e_j}$$

where L_{e_i} and R_{e_j} are left and right Clifford multiplications, we have the formula

$$D_s^2 = D^2 + s^2 |dh|^2 + s\mathbf{H}_h.$$

The Witten complex

Some useful properties of D carry over to the perturbed version D_s:

- D_s belongs to the class of self-adjoin generalized Dirac operators, thus the basic elliptic theory of Chapter 5 applies.
- Finite propagation speed also holds with the same proof for the standard Dirac operator in Chapter 7.

Proposition

The Garding inequality holds for D_s, i.e. there exists a constant $C > 0$ such that for any $w \in C^\infty(S)$,

$$||w||_1 \leq C(||w||_0 + ||D_s w||_0)$$

where $||\cdot||_k$ denotes the Sobolev-k norm.

Proof. Write $D_s^2 = D^2 + s\mathbf{H}_h + s^2|dh|^2 = D^2 + L$ where L is of order zero. Therefore

$$||D_s w||^2 = ||Dw||^2 + \langle Lw, w\rangle \geq ||Dw||^2 - C_1||w||^2$$

for some constant C_1. Hence

$$(1 + C_1)(||D_s w||^2 + ||w||^2) \geq ||Dw||^2 + ||w||^2 \geq \frac{1}{C_2}||w||_1^2,$$

where the second inequality follows from the usual Garding inequality. □

Remark. The norm of L is of order s^2, so the constant C_1 is bounded by a polynomial in s. Analogously, so are the constants in the elliptic estimates bounded by polynomials in s.

Perturbed Morse Inequalities

Proposition

Consider the Betti numbers of the generalized Dirac complex (S, d),

$$\beta_j{}^s = \dim H^j(S, d),$$

and for a smooth rapidly decreasing positive function φ with $\varphi(0) = 1$, set

$$\mu_j = \operatorname{Tr}(\varphi(D^2)\big|_{S_j}).$$

Then

$$\sum_{j=1}^{k} (-1)^j \beta_j{}^s \leq \sum_{j=1}^{k} (-1)^j \mu_j$$

with equality when $k = \dim M$.

- To deduce the standard Morse inequalities, we take the family D_s above, and investigate the perturbed inequalities as $s \to \infty$.

Perturbed Morse Inequalities

Proof. We want to show

$$\sum_{j=1}^{k}(-1)^j(\mu_j - {\beta_j}^s) \geq 0$$

with equality holding when k is of top dimension.

By Hodge theory in Chapter 6,

$${\beta_j}^s = \dim ker\, D^2.$$

Since D^2 has discrete spectrum, there is a smooth, positive, rapidly decreasing function $\tilde{\varphi}$ on $\mathbb{R}^+$ such that $\tilde{\varphi}(0) = 1$ and $\tilde{\varphi}(\lambda) = 0$ on all of the eigenvalues of D^2. Therefore,

$${\beta_j}^s = \mathrm{Tr}(\tilde{\varphi}(D^2)|_{S_j}).$$

Perturbed Morse Inequalities

Now we may assume that $\varphi \geq \tilde{\varphi}$. Then

$$\mu_j - {\beta_j}^s = \mathrm{Tr}((\varphi - \tilde{\varphi})(D^2)|_{S_j}).$$

For a positive, rapidly decreasing function ψ, vanishing and differentiable at 0, we may write

$$(\varphi - \tilde{\varphi})(\lambda) = \lambda(\psi(\lambda))^2$$

By functional calculus, we may write

$$(\varphi - \tilde{\varphi})(D^2) = D^2(\psi(D^2))^2.$$

Now $D^2 = dd^* + d^*d$, using the trace argument in Chapter 10,

$$\begin{aligned} \mathrm{Tr}(dd^*(\varphi(D^2))^2|_{S_j}) &= \mathrm{Tr}(d^*(\varphi(D^2))^2|_{S_j} d) \\ &= \mathrm{Tr}(d^*d(\varphi(D^2))^2|_{S_{j-1}}). \end{aligned}$$

Perturbed Morse Inequalities

Therefore,

$$\begin{aligned}\sum_{j=1}^{k}(-1)^j(\mu_j - {\beta_j}^s) &= (\mu_j - {\beta_j}^s) - (\mu_{j-1} - {\beta_{j-1}}^s) - \cdots \\ &= \mathrm{Tr}((dd^* + d^*d)(\varphi(D^2))^2|_{S_j}) - \cdots \\ &= \mathrm{Tr}(d^*d(\varphi(D^2))^2|_{S_j}) + \mathrm{Tr}(d^*d(\varphi(D^2))^2|_{S_{j-1}}) - \cdots \\ &= \mathrm{Tr}(d^*d(\varphi(D^2))^2|_{S_j})\end{aligned}$$

The last equality follows since the middle terms cancel pairwise and the last term

$$\mathrm{Tr}(dd^*(\varphi(D^2))^2|_{S_0}) = 0$$

In particular, when j is of top dimension, the expression vanishes, yielding the desired equality.

Perturbed Morse Inequalities

Meanwhile, we may write for $A = d\varphi(D^2)|_{S_j}$,

$$\mathrm{Tr}(d^* d(\varphi(D^2))^2|_{S_j}) = \mathrm{Tr}(A^* A)$$

which is positive, since for any orthonormal basis, we may compute

$$\mathrm{Tr}(A^* A) = \sum_i \langle A^* A e_i, e_i \rangle = \sum ||Ae_i||^2 \geq 0.$$

This gives

$$\sum_{j=1}^{k} (-1)^j (\mu_j - {\beta_j}^s) = \mathrm{Tr}(d^* d(\varphi(D^2))^2|_{S_j}) \geq 0.$$

as desired, with equality in top dimension. $\square$

Asymptotic calculations

We first investigate the asymptotics of $\varphi(D_s)$ as $s \to \infty$ outside the critical points of h.

Fix a number $\rho > 0$, and choose a positive even function $\varphi \in \mathcal{S}(\mathbb{R})$ with $\varphi(0) = 1$ such that the Fourier transform $\hat{\varphi}$ is supported within $[\rho, -\rho]$.

Lemma

On the complement of a 2ρ-neighborhood of Crit(h) the smoothing kernel $k_s(p,q)$ on $M \times M$ of $\varphi(D_s)$ tends uniformly to zero as $s \to \infty$.

Proof.

- Existence of self-adjoint extension A:

  Since M is compact, there is a constant $C > 0$ such that $|\nabla h(x)| \geq C$ for all x in the complement of a ρ-neighborhood of Crit(h).

Asymptotic calculations

By the formula

$$D_s^2 = D^2 + s\mathbf{H}_h + s^2|dh|^2$$

we find that for s large

$$\langle D_s^2\omega,\, \omega\rangle \geq \frac{1}{2}C^2 s^2 ||\omega||^2.$$

Let $\mathscr{H}$ denote the Hilbert space of L^2 differential forms on M that vanish on a ρ-neighborhood of $\mathrm{Crit}(h)$. Then D_s^2 is a positive formally self-adjoint (symmetric) operator on $\mathscr{H}$.

By Friedrichs' extension theorem, there exists a self-adjoint extension A on $\mathscr{H}$.

- Claim: If ω is supported in the complement of a 2ρ-neighborhood of $\mathrm{Crit}(h)$, then

$$\varphi(D_s)\omega = \varphi(\sqrt{A})\omega.$$

Asymptotic calculations

Consider the time dependent differential form

$$\omega_t = \cos(tD_s)\omega = \frac{1}{2}(e^{itD_s} + e^{-itD_s})\omega$$

which is the unique solution to the partial differential equation

$$\frac{\partial^2 \omega_t}{\partial t^2} + D_s^2 \omega_t = 0$$

with the initial conditions $\omega_0 = \omega$, $\dot{\omega_0} = 0$.

By the unit propagation speed property, ω_t is supported in the complement of a ρ-neighborhood of $\mathrm{Crit}(h)$ for $|t| < \rho$. Therefore $D_s^2 \omega_t = A\omega_t$. Thus ω_t for $|t| < \rho$ is also the unique solution to the equation with the same initial conditions

$$\frac{\partial^2 \omega_t}{\partial t^2} + A\omega_t = 0.$$

Asymptotic calculations

We thus write $\omega_t = \cos(t\sqrt{A})\omega$. We compute

$$\begin{aligned}
\varphi(D_s)\omega &= \frac{1}{2\pi}\int_{-\rho}^{\rho}(e^{itD_s})\hat{\varphi}(t)\,dt \\
&= \frac{1}{\pi}\int_0^{\rho}\hat{\varphi}(t)\cos(tD_s)\omega\,dt \\
&= \frac{1}{\pi}\int_0^{\rho}\hat{\varphi}(t)\omega_t\,dt \\
&= \frac{1}{\pi}\int_0^{\rho}\hat{\varphi}(t)\;\cos(t\sqrt{A})\omega\,dt \\
&= \varphi(\sqrt{A})\omega.
\end{aligned}$$

This proves our claim.

Asymptotic calculations

Now $\sqrt{A}$ is positive. In fact, $\sqrt{A} \geq Cs/2$ as a result of the inequality

$$\langle D_s^2 \omega, \omega \rangle \geq \frac{1}{2} C^2 s^2 ||\omega||^2.$$

By the spectral theorem, the operator norm of $\varphi(\sqrt{A})$ is bounded above by

$$c(s) = \sup \{ |\varphi(\lambda)| : \lambda \geq \frac{1}{2} Cs \}$$

That is,

$$||\varphi(D_s) w|| \leq c(s) ||w||$$

For w supported outside a complement of a 2ρ-neighborhood of the critical points. But since φ is rapidly decreasing, we have that $c(s) \to 0$ rapidly as $s \to \infty$.

This is very close to what we want for the kernel.

Asymptotic calculations

- Claim: There is a $c_1(s) \to 0$ as $s \to \infty$, with

$$||\varphi(D_s)\omega||_{L^\infty} \leq c_1(s)||\omega||_{L^1}.$$

Elliptic estimates: for any k, the operator $(1 + D_s^2)^{-1}$ is bounded as an operator from W^k to W^{k+2} with norm bounded by polynomials in s.

Sobolev embedding: $W^p \subset L^\infty$ for $p > n/2$. Therefore $(1 + D_s)^{-k}$ is bounded from L^2 to L^∞ for $k > n/4$. By duality and self-adjointness $(1 + D_s)^{-k}$ is also bounded as an operator from L^1 to L^2, with both bounds being polynomial in s.

Therefore, $\varphi(D_s)$ acting as an operator from L^1 to L^∞ is bounded by a polynomial in S times the norm of

$$\varphi'(D_s) = (1 + D_s^2)^{2k}\phi(D_s),$$

where $\varphi'(\lambda) = (1 + \lambda^2)^{2k}\varphi(\lambda)$.

Asymptotic calculations

But since φ' satisfies the same conditions as φ, we have the estimate

$$||\varphi'(D_s)w|| \leq c'(s)||w||.$$

with $c'(s)$ rapidly decaying in s.

Combining these results, we have the bound

$$||\varphi(D_s)w||_{L^\infty} \leq c_1(s)||w||_{L^1}$$

where $c_1(s) = c'(s) \times$ (polynomial in s), and tends to zero as $s \to \infty$. This proves the lemma, since the supremum of the kernel can be estimated by the left and side of the claim.

Remark. This result suggests that as $s \to \infty$, $\mathrm{Tr}(\varphi(D_s))$ is given by a sum of contributions from the critical points of h. This quantity becomes useful when considering the perturbed Morse inequalities.

Contribution from critical points

- Choose a metric g on M as follows: Let g be flat Euclidean $g_{ij} = \delta_{ij}$ in Morse coordinates near each critical point, and patch these metrics away from $\mathrm{Crit}(h)$ via a partition of unity.
- Choose ρ small such that g is flat Euclidean at least a distance 4ρ from each critical point.
- Then by a previous calculation in Chapter 9, we have

$$D_s^2 := L_s = \sum_j -\left(\frac{\partial}{\partial x_j}\right)^2 + s^2\lambda_j^2 x_j^2 + s\lambda_j Z_j$$

  where $Z_j = [dx_j \lrcorner, dx_j \wedge]$ and λ_j supplied by the Morse lemma.
- Furthermore, L_s has eigenvalues for $p_j = 0, 1, 2, \ldots$ and $q_j = \pm 1$.

$$s\sum_j (|\lambda_j|(1 + 2p_j) + \lambda_j q_j)$$

  Restricted to k-forms, the spectrum has the additional property that exactly k of the q_j's are equal to $+1$.

Contribution from critical points

Now let φ be as in the previous lemma with $\varphi(0) = 1$. Then $\varphi(D_s) = \varphi(\sqrt{L_s})$ is smoothing and hence of trace class. (Thm 8.10)

Lemma

Suppose that the first m of the λ_j's are negative. We have (i):

$$\lim_{s\to\infty} \mathrm{Tr}\left(\varphi(\sqrt{L_s})\big|_{\bigwedge^k}\right) = \begin{cases} 1 & (k \neq m) \\ 0 & (k = m) \end{cases} .$$

(ii) Furthermore, the same limit holds for

$$\lim_{s\to\infty} \mathrm{Tr}\left(B\varphi(\sqrt{L_s})\big|_{\bigwedge^k}\right) = \begin{cases} 1 & (k \neq m) \\ 0 & (k = m) \end{cases} .$$

where B is the multiplication operator on $\mathbb{R}^n$ by some $\beta \in C_c^\infty(\mathbb{R}^n)$ with $\beta(0) = 1$.

Contribution from critical points

Proof. We have

$$\mathrm{Tr}\left(\varphi(\sqrt{L_s})\big|_{\bigwedge^k}\right) = \sum_{p_j,q_j} \varphi\left(\sqrt{s\sum_j (|\lambda_j|(1+2p_j)+\lambda_j q_j)}\right)$$

(i) If $k \neq m$ then all the eigenvalues of L_s are of order s. Since φ is rapidly decreasing, the sum tends to 0 as $s \to \infty$.

Meanwhile, if $k = m$, then precisely one eigenvalue is 0 while the others are of order s. So

$$\mathrm{Tr}\left(\varphi(\sqrt{L_s})\big|_{\bigwedge^k}\right) = 1 + \sum_{\text{rest of } p,q} \varphi(M_{p,q}\sqrt{s})$$

for constants $M_{p,q} > 0$ which tends to 1 as $s \to \infty$.

Contribution from critical points

(ii) Let $e(p_j, q_j)$ be normalized eigenforms of L_s of (p_j, q_j). Then

$$\operatorname{Tr}\left(B\varphi(\sqrt{L_s})\big|_{\bigwedge^k}\right) =$$

$$\sum_{p_j, q_j} \varphi\left(\sqrt{s\sum_j (|\lambda_j|(1+2p_j) + \lambda_j q_j)}\right) \langle Be(p_j, q_j),\, e(p_j, q_j)\rangle$$

Only the zero eigenvalue contribute, giving the value $\langle Be_0, e_0\rangle$ in the limit $s \to \infty$, where e_0 is the eigenform corresponding to eigenvalue 0.

Now e_0 is the ground state of the harmonic oscillator multiplied by $dx^1 \wedge \cdots \wedge dx^k$. Explicitly,

$$e_0 = (s^{n/2} f(\lambda_j)) \exp(-sg(\lambda_j) dx^1 \wedge \cdots \wedge dx^k)$$

for some functions f and g. In particular, as $s \to \infty$, $\langle Be_0,\, e_0\rangle \to 1$, which proves the second statement. $\square$

Proof of the Theorem

Proof. (Morse inequalities)

- Let g metric on M, $\rho > 0$ and φ positive even function be as described above such that the previous results (asymptotic calculation, contribution from critical points) are applicable.
- Moreover, note that since φ is even, we can write $\varphi(D_s) = \psi(D_s^2)$ for some function ψ, so the perturbed Morse inequalities also hold for

$$\beta_j{}^s = \dim H^j(S, d_s), \quad \mu_j = \mathrm{Tr}(\varphi(D_s)\big|_{\bigwedge^k T^*M})$$

- Claim 1: $\beta_j{}^s = \beta_j$.

  Recall the latter is the dimension of the standard de Rham cohomology. The perturbed dimension is unchanged due to the nature of the perturbation.

Proof of the Theorem

- Claim 2: ${\mu_j}^s \to \nu_j$ as $s \to \infty$.

  Recall ν_j is the number of critical points of index j. Now

  $$\mathrm{Tr}(D_s)\big|_{\bigwedge^j} = \int \mathrm{tr}\, k_s(m,m)\, dvol_M,$$

  where the trace is taken over $k(m,m) \in S_m \otimes S_m^*$. By our asymptotic calculation $k_s(m,m) \to 0$ uniformly as $s \to \infty$ outside a neighborhood of critical points.

  So the only contribution to the trace is from critical points, which for some $p \in \mathrm{Crit}(h)$ could be written as

  $$\lim_{s\to\infty} \mathrm{Tr}(B\varphi(D_s)\big|_{\wedge^j})$$

  for B multiplication by a smooth function $\beta \equiv 1$ on a 2ρ-nbhd of p and $\mathrm{supp}(\beta)$ is contained in a 3ρ-nbhd of p.

Proof of the Theorem

Taking Morse coordinates around p, we can identify a 4ρ-nbhd of p in M with a 4ρ-nbhd of $\mathbb{R}^n$. Under this identification $D_s^2 = L_s$ as in the results before. By an analogous unit propagation speed argument to that of the asymptotic calculations, we have

$$\varphi(D_s)w = \varphi(\sqrt{L_s})w$$

for w supported in a 3ρ-neighborhood of p. Hence

$$\mathrm{Tr}(B\varphi(D_s)\big|_{\bigwedge^j}) = \mathrm{Tr}(B\varphi(\sqrt{L_s})\big|_{\bigwedge^j})$$

which as $s \to \infty$, tends to 1 if the critical point has index j, and zero otherwise. $\square$