Applications of the Atiyah-Singer-Index Theorem

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Statement of the Theorem

Theorem (Atiyah-Singer-Index)

Let M be a closed, oriented, smooth, even-dimensional manifold, and S a canonically graded Clifford bundle over M with associated Dirac operator D. Then

$$Ind(D) = \langle\, \hat{\mathcal{A}}(TM)\, ch(S/\Delta),\, [M]\,\rangle,$$

where $ch(S/\Delta)$ is the relative Chern character of S.

- The theorem asserts equality of the **analytical index** of D and the **topological index** based on topological data from M and S.
- We will examine the theorem on four different (generalized) Dirac operators and obtain several known results:
  - Hirzebruch Signature Theorem
  - Chern-Gauss-Bonnet Theorem
  - Hirzebruch-Riemann-Roch Theorem

Analytical Index

Definition (Grading operator)

A **grading operator** $\epsilon : S \to S$ is a self-adjoint involution that commutes with the covariant differentiation (parallel), and anti-commutes with Clifford multiplication of vectors, i.e.

$$\epsilon^2 = 1, \quad [\epsilon, \nabla] = 0, \quad \{\epsilon, c(v)\} = 0.$$

Definition (Graded Clifford bundle)

A **graded Clifford bundle** S is one provided with a grading operator ϵ the decomposes $S = S_+ \oplus S_-$ into its ± 1 eigenspaces $S_\pm$.

Hence, the Dirac operator anticommutes with the grading operator,

$$D_\pm : C^\infty(S_\pm) \to C^\infty(S_\mp)$$

The **Index of** D is defined to be

$$\mathrm{Ind}\,(D) = \dim \mathit{ker}\, D_+ - \dim \mathit{ker}\, D_-.$$

Analytical Index

Definition ((Anti-)Canonically graded Clifford bundle)

Let $\epsilon_0 := i^m \omega$ denote the canonical grading on an even-dimensional, oriented manifold, where ω is the volume element of $Cl(k)$. A grading of S is **canonical** if $\epsilon = \epsilon_0$, and **anti-canonical** if $\epsilon = -\epsilon_0$.

Lemma

Any graded Clifford bundle S splits into a direct sum of the canonically graded and anti-canonically graded parts $S = S_C \oplus S_A$.

- Note: This splitting need not have anything to do with the splitting $S = S_+ \oplus S_-$.
- Question: What happens to the Index Theorem when the grading is not canonical?

Spin Representation and Spin Bundles

For M a Spin manifold of even-dimension $2m$, recall the following facts:

- There is a unique irreducible complex $Cl(k)$-representation Δ of dimension 2^m called the **Spin representation**.
- Any other $Cl(k)$-representation is given by $W = \Delta \otimes V$, where V is an auxiliary coefficient vector space.
- $\Delta = \Delta_+ \oplus \Delta_-$ as a $Spin(k)$-representation splits into a direct sum of ± 1 eigenspaces of $i^m \omega$ (i.e. a graded representation of $Cl(k)$).
- The **Spin bundle** (also denoted) Δ is associated to the spin structure on M via the spin representation, and has the canonical grading $\Delta = \Delta_+ \oplus \Delta_-$.
- Any Clifford bundle on M has form $S = \Delta \otimes V$ for some vector bundle $V = \mathrm{Hom}_{Cl}(\Delta, S)$.

Topological Index

Definition (Chern f-genus)

Let E be a complex vector bundle over M and $h^*E = \oplus_i L_i$ for $h : X \to M$ given by the splitting principle, with first Chern classes $c_1(L_i) = x_i$. The **Chern f-genus** is defined for f holomorphic near 0, $f(0) = 1$

$$\Pi_f(E) = \prod_i f(x_i)$$

Definition (Pontryagin g-genus)

Let V be a real vector bundle over M and $h^*(V \otimes \mathbb{C}) = \oplus_i K_i$ for $h : Y \to M$ given by the splitting principle, with first Pontryagin classes $p_1(K_i) = y_i$. The **Pontryagin g-genus** is defined for g holomorphic near 0, $g(0) = 1$,

$$\mathcal{G}_g(V) = \prod_i g(y_i)$$

Examples

- The total Chern class is the Chern f-genus of $f(z) = 1 + z$. Consider $X = \oplus_i L_i$ with $c_1(L_i) = x_i$, then

$$\begin{aligned} c(X) &= 1 + c_1(X) + c_2(X) + \cdots + c_n(X) \\ &= (1 + x_1) \cdots (1 + x_n) = \prod_i (1 + x_i) = \Pi_f(X) \end{aligned}$$

- The $\hat{\mathcal{A}}$**-genus** of a vector bundle V is the Pontryagin genus of

$$g(z) = \sqrt{z}/2 \Big/ \sinh \sqrt{z}/2.$$

Proposition

Let V be a complex vector bundle, and g a suitable function. The Pontryagin g-genus is the Chern f-genus for $f(z) = g(z^2)^{1/2}$.

Topological Index

Definition (Chern character)

Given a complex vector bundle V and chern roots $x_1, ..., x_n$, the **Chern character** is the characteristic class

$$\mathrm{ch}(V) = \sum e^{x_i}.$$

Proposition

Let M be a $2m$-dimensional spin manifold, and Δ the associated spin bundle. Let $g(z) = \cosh(\sqrt{z}/2)$. Then Chern character $\mathrm{ch}(\Delta)$ is given

$$\mathrm{ch}(\Delta) = 2^m \mathcal{G}_g(TM).$$

Topological Index

Definition (Relative Chern character)

Let S be a Clifford bundle on M. The **relative Chern character** of S is defined as

$$\mathrm{ch}(S/\Delta) = \mathrm{tr}^{S/\Delta}(\exp(-F^S/2\pi i)).$$

where F^S is the twisting curvature of S, and $\mathrm{tr}^{S/\Delta}$ is the relative trace.

- It is convenient to compute the relative Chern character locally where $S \cong \Delta \otimes V$ for some vector bundle V.
- In this case,

$$K^S = K^\Delta \otimes 1 + 1 \otimes K^V$$

  so the twisting curvature is simply the curvature of V, $F^S = K^V$, and the relative Chern character is given by

$$\mathrm{ch}(S/\Delta) = \mathrm{tr}^V(\exp(-K^V/2\pi i)) = \mathrm{ch}(V).$$

Application: Dirac Operator

Let M be a closed k-dimensional spin manifold, $k = 2m$. We consider the Dirac complex for a general Clifford bundle S on M:

- Clifford bundle: $S = \Delta \otimes V$
- Grading: $\epsilon = i^m \omega$ (Canonical)
- Dirac operator: $D_A = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}$
- Graded structure: $S = (\Delta_+ \otimes V) \oplus (\Delta_- \otimes V)$

Corollary

The Index Theorem yields

$$Ind\,(D_A) = \langle\, \hat{\mathcal{A}}(TM)\, ch(V),\, [M]\,\rangle.$$

In particular, if M is 4-dimensional and S is the spin bundle,

$$Ind\,(D_A) = \langle\, \hat{\mathcal{A}}(TM),\, [M]\,\rangle = -\frac{1}{24} p_1(M).$$

Application: Dirac Operator

Corollary

If M is 4-dimensional and S is associated to a Spin^c-structure,

$$Ind(D_A) = \langle \hat{\mathcal{A}}(TM) \cdot e^{c_1(L_S/2)}, [M] \rangle,$$

where L_S is the characteristic line bundle of S. (cf. MGT notes p.57)

Proof. There is a line bundle E such that $S = \Delta \otimes E$ and

$$2c_1(E) + c_1(L_\Delta) = c_1(L_S)$$

Since Δ is a spin structure $c_1(L_\Delta) = 0$ and

$$\mathrm{ch}(E) = e^{c_1(E)} = e^{c_1(L_S)/2}.$$

□

Application: Dirac Operator

Theorem (Lichnerowicz)

Let M be a closed spin manifold admitting a positive scalar curvature. Then $\hat{\mathcal{A}}(M) = 0$. (cf. MGT notes p.61 for an alternative proof)

Proof. For the spin bundle, the Weitzenböck formula yields

$$D^2 = \nabla^* \nabla + \frac{1}{4}\kappa.$$

Thus if $\kappa > 0$, $ker\, D = ker\, D^2 = 0$. By the Index Theorem

$$0 = \dim ker\, D_+ - \dim ker\, D_- = \mathrm{Ind}\,(D) = \langle \hat{\mathcal{A}}(TM),\, [M]\,\rangle,$$

that is, $\hat{\mathcal{A}}(M) = 0$. □

Application: Signature Operator

Let M be a closed $4k$-dimensional Riemmanian manifold, $4k = 2m$. We consider the following generalized Dirac complex on M:

- Clifford bundle: $S = \bigwedge^* T^*M \otimes \mathbb{C}$
- Grading: $\epsilon = i^m \omega$ (Canonical)
- Signature operator: $D = d + d^*$

Lemma

The grading operator could be expressed in terms of the Hodge operator

$$\epsilon = i^m \omega = i^{m+p(p-1)} * \text{ (on } p\text{-forms).}$$

Proof. Use the fact that $\omega v = (-1)^{k-1} v\omega$ for vectors, and the reversal operator

$$a_1 \ldots a_p = (-1)^{p(p-1)/2} a_p \ldots a_1 = i^{p(p-1)} a_p \ldots a_1.$$

□

Application: Signature Operator

We thus have $\epsilon = *$ (on m-forms), $\epsilon = (-1)^{k-1}*$ (on p-forms, $p \neq m$).

- Graded structure: $S = S_+ \oplus S_-$, depending on k odd or even

$$S_+ = \bigoplus_{p \neq m} \mathcal{H}_p^{\pm} \oplus \mathcal{H}_m^{+}, \quad S_- = \bigoplus_{p \neq m} \mathcal{H}_p^{\mp} \oplus \mathcal{H}_m^{-}$$

Proposition

For the signature operator D of the generalized Dirac complex above,

$$\mathrm{Ind}\,(D) = \sigma(M)$$

where $\sigma(M)$ defined by the intersection form on $H^{2k}(M;\mathbb{R})$.

Proof. Since $ker\, D = ker\, D^2 = ker\, \Delta$, we may split the Laplacian using the canonical grading $\Delta = \Delta^+ \oplus \Delta^-$,

$$\mathrm{Ind}\,(D) = \dim ker\, \Delta^+ - \dim ker\, \Delta^-.$$

Application: Signature Operator

- Claim: Restricted to $\mathcal{H}_p^{\pm}$, $p \neq m$,

$$\dim ker\, \Delta^{+} = \dim ker\, \Delta^{-}.$$

For let $0 \neq \beta \in ker\, \Delta$. Then $\alpha = \beta + \epsilon(\beta) \in ker\, \Delta^{+}_{p \neq m}$. But immediately, we have a corresponding nonzero element

$$\alpha' = \beta - \epsilon(\beta) \in ker\, \Delta^{-}_{p \neq m},$$

which proves the claim. Therefore,

$$\begin{aligned} \text{Ind}\,(D) &= \dim ker\, \Delta^{+}_{m} - \dim ker\, \Delta^{-}_{m} \\ &= \dim \mathcal{H}^{+}_{m} - \dim \mathcal{H}^{-}_{m} = \sigma(M). \end{aligned}$$

The second equality follows from the construction $\alpha = \beta \pm \epsilon(\beta)$ as in the claim. $\square$

Application: Signature Operator

Theorem (Hirzebruch Signature Theorem)

For a closed, oriented, smooth, $4k$-dimensional manifold M,

$$\sigma(M) = \langle \mathcal{L}(TM), [M] \rangle,$$

where $\mathcal{L}$-genus is the Pontryagin genus associated to

$$g(z) = \sqrt{z} \big/ \tanh \sqrt{z}.$$

Proof. Since Δ is a irreducible representation, we have locally

$$\bigwedge^{*} T^{*}M \otimes \mathbb{C} \cong Cl(k) \otimes \mathbb{C} \cong End(\Delta) \cong \Delta \otimes \Delta^{*} \cong \Delta \otimes \Delta$$

We can compute the relative Chern class

$$\mathrm{ch}(S/\Delta) = \mathrm{ch}(\Delta) = 2^{m} \mathcal{G}_{g}(TM).$$

Application: Signature Operator

Thus the Index Theorem tells us that

$$\sigma(M) = 2^m \langle \hat{\mathcal{A}}(TM)\, \mathcal{G}_g(TM),\, [M] \rangle.$$

Now on the right hand side

$$\begin{aligned} 2^m \hat{\mathcal{A}}(TM)\, \mathcal{G}_g(TM) &= 2^m \prod_{i=1}^{m} \frac{\sqrt{y_i}/2}{\sinh \sqrt{y_i}/2} \cdot \cosh \sqrt{y_i}/2 \\ &= 2^m \prod_{i=1}^{m} \frac{\sqrt{y_i}/2}{\tanh \sqrt{y_i}/2} \end{aligned}$$

Focusing on the $2m$-dimensional coefficient, as is the only relevant contribution,

$$2^m \left[\prod_{i=1}^{m} \frac{\sqrt{y_i}/2}{\tanh \sqrt{y_i}/2} \right]_{2m} = \left[\prod_{i=1}^{m} \frac{\sqrt{y_i}}{\tanh \sqrt{y_i}} \right]_{2m} = [\mathcal{L}(TM)]_{2m}.$$

□

Topological Index: De Rham Operator

Let M be a closed k-dimensional Riemmanian manifold, $k = 2m$. We consider the following generalized Dirac complex on M:

- Clifford bundle: $S = \bigwedge^* T^*M \otimes \mathbb{C}$
- Grading: $\epsilon = (-1)^q$ on (q-forms)
- Signature operator: $D = d + d^*$
- Graded structure: $S = \bigwedge^*_{\text{even}} T^*M \otimes \mathbb{C} \oplus \bigwedge^*_{\text{odd}} T^*M \otimes \mathbb{C}$

Note that the Euler grading is neither canonical nor anti-canonical. In this case,

$$\begin{aligned} \operatorname{Ind}(D) &= \dim ker\, d + d^*|_{\text{even}} - \dim ker\, d + d^*|_{\text{odd}} \\ &= \dim \mathcal{H}_{k,\text{even}} - \dim \mathcal{H}_{k,\text{odd}} \\ &= (-1)^k \dim H^k(M;\mathbb{R}) = \chi(M) \end{aligned}$$

where the third equality follows from Hodge theory on Riemannian manifolds.

Application: De Rham Operator

Corollary (Chern-Gauss-Bonnet Theorem)

The Index Theorem yields

$$\chi(M) = \langle e(TM), [M] \rangle$$

which, coupled with the Gauss-Bonnet Theorem, gives the Chern-Gauss Bonnet Theorem

$$\frac{1}{4\pi} \int \kappa \, dvol = \langle e(TM), [M] \rangle.$$

Proof. First: split S into its canonically and anticanonically graded parts. As before, locally $S = \Delta \otimes \Delta$. Suppose for $\Delta = V \oplus W$,

$$S_C = (\Delta_+ \otimes V) \oplus (\Delta_- \otimes V), \quad S_A = (\Delta_- \otimes W) \oplus (\Delta_+ \otimes W)$$

Then since $\Delta = \Delta_+ \oplus \Delta_-$, we see that $V = \Delta_+$, $W = \Delta_-$.

Application: De Rham Operator

Hence

$$S_C = \Delta \otimes \Delta_+, \quad S_A = \Delta \otimes \Delta_-.$$

Next, we compute the super relative Chern character

$$\begin{aligned} \mathrm{ch}_s(S/\Delta) &:= \mathrm{ch}(S_C/\Delta) - \mathrm{ch}(S_A/\Delta) \\ &= \mathrm{ch}(\Delta_+) - \mathrm{ch}(\Delta_-) \\ &= 2^m \, \mathcal{G}_h(TM) \end{aligned}$$

where $h(z) = \sinh \sqrt{z}/2$. The last equality could be computed in the 2-dimensional case by

$$e^x - e^{-x} = 2 \sinh x$$

for $x = c_1(\Delta_+)$. The arguments are exactly analogous to the computation of $\mathrm{ch}(\Delta)$.

Application: De Rham Operator

Finally, putting together the ingredients, the Index Theorem implies

$$\begin{aligned}\chi(M) &= \langle \hat{\mathcal{A}}(TM)\,\mathrm{ch}_s(S/\Delta)\,,[M]\rangle \\ &= 2^m\langle \hat{\mathcal{A}}(TM)\,\mathcal{G}_h(TM)\,,[M]\rangle.\end{aligned}$$

Now on the left hand side

$$\begin{aligned}2^m\,\hat{\mathcal{A}}(TM)\mathcal{G}_h(TM) &= 2^m\prod_{i=1}^m \frac{\sqrt{y_i}/2}{\sinh\sqrt{y_i}/2}\cdot\sinh\sqrt{y_i}/2 \\ &= 2^m\prod_{i=1}^m \sqrt{y_i}/2 = \prod_{i=1}^m \sqrt{y_i} = e(TM).\end{aligned}$$

The last equality follows from Question 2.36, p.39 of Roe. □

Application: Dolbeault Operator

Let V be an inner product space with complex structure J compatible with the metric. One can decompose

$$V \otimes \mathbb{C} = P \oplus Q$$

into its $\pm i$ eigenspaces of J. Recall that $\bigwedge^* P = \bigwedge^* \bar{Q}$ carries the spin representation. It therefore makes sense to consider this construction on complex manifolds.

Let M be a closed k-dimensional Kähler manifold and W a holomorphic vector bundle over M. We consider the spin representation associated to a $Spin^c$-structure on M:

- Clifford bundle: $S = \bigwedge^*(T^{0,1}M)^{*c} \otimes W = \Delta^c \otimes W$
- Grading: $(-1)^q$ (on $(0,q)$-forms)
- Dolbeault operator: $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$
- Graded structure: $(\Delta^c_+ \otimes W) \oplus (\Delta^c_- \otimes W)$

Application: Dolbeault operator

Lemma

The $\hat{\mathcal{A}}$-genus of TM is equal to the Chern f-genus of $T^{1,0}M$ with

$$f(z) = \frac{z/2}{\sinh z/2}.$$

Proof. By the correspondence between the Chern f-genus and the Pontryagin g-genus, we can write $\hat{\mathcal{A}}(TM)$ as $\Pi_{\tilde{f}}(TM)$ where

$$\tilde{f}(z) = \sqrt{g(z^2)} = \left(\frac{\sqrt{z^2}/2}{\sinh \sqrt{z^2}/2}\right)^{1/2}.$$

But since $TM = T^{1,0}M \oplus T^{0,1}M$, and $c_i(T^{1,0}M) = (-1)^i c_i(T^{0,1}M)$,

$$\Pi_{\tilde{f}}(TM) = \Pi_{\tilde{f}}(T^{1,0}M)\Pi_{\tilde{f}}(T^{0,1}M) = \Pi_f(T^{1,0}M)$$

for the given f above. □

Application: Dolbeault operator

Corollary (Hirzebruch-Riemann-Roch Theorem)

On the generalized Dirac complex as the above, the Index Theorem yields

$$\sum_k (-1)^k \, dim\, H^{0,k}(W) = \langle Td(T^{1,0}M)\, ch(W), [M]\rangle,$$

where the ***Todd-genus*** *Td of a complex vector bundle is by definition the Chern genus associated to $f(z) = z/(e^z - 1)$.*

Proof. For the right hand side we compute

$$\hat{\mathcal{A}}(TM)\,\mathrm{ch}(S/\Delta) = \Pi_f(T^{1,0}M)\, e^{c_1(L_S)/2}\,\mathrm{ch}(W)$$

In terms of functions associated to the Chern genus of $T^{1,0}M$, the first two products yield the Todd genus of $T^{1,0}M$

$$\frac{z/2}{\sinh z/2} \cdot e^{-z/2} = \frac{z}{e^z - 1}.$$

□

Remarks

- The Kähler condition could be relaxed with the replacement $\sqrt{2}(\bar{\partial} + \bar{\partial}^*) = D + A$, where $A \in End(S)$ is a zero order term. (cf. Theorem 13.13, p.77, Roe)
- The Index Theorem is a more general result of which several known Theorems occur as special cases.
- A historically earlier version of the Atiyah-Singer-Index Theorem takes the form

$$\mathrm{Ind}\,(D) = \langle \mathrm{ch}(\sigma(D))Td(TM)\,, [M]\rangle$$

  which has a term directly involving the symbol of D and could be proved by using K-theory.
- The Hirzebruch-Riemann-Roch Theorem is a very important tool in algebraic geometry. When restricted to divisor line bundles, one obtains as a special case the Riemann-Roch Theorem

$$\dim H^0(X, D) - \dim H^1(X, D) = \deg D + 1 - g(X).$$