Differentiable manifolds
WS 2015/16

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13 October 2015

(1) Topological spaces; Hausdorffness; continuity.
(2) Metric spaces, and their induced topology. This topology is always Hausdorff.
(3) Euclidean space $\mathbb{R}^n$ with its metric topology. By considering balls with rational radii centered at points with rational coordinates, one finds a countable collection of open sets such that all open sets are suitable unions of these.
(4) Bases for topologies. Remark that by the previous item, Euclidean space has a countable bases for its topology.
(5) **Topological manifolds** (of dimension n) are topological spaces that are locally homeomorphic to $\mathbb{R}^n$, are Hausdorff, and have a countable basis for their topology. The latter condition is a weakening of compactness that will allow us to make inductive constructions to pass from local to global statements.

15 October 2015

(6) A **differentiable manifold** is a topological manifold together with an atlas whose transition maps are differentiable. Such an atlas is called a differentiable or smooth atlas.
(7) A **differentiable structure** is an equivalence class of atlases, equivalently a maximal atlas.
(8) Every maximal C^r atlas with $r \geq 1$ contains a C^∞ atlas. Because of this fact (which we do not prove), we will restrict ourselves to C^∞ manifolds throughout. So the words differentiable or smooth will usually mean C^∞.
(9) We define differentiability for maps between differentiable manifolds using charts.
(10) We will consider two differentiable manifolds to be the same if there is a **diffeomorphism** between them, i.e. a differentiable bijection with differentiable inverse. Since differentiablity implies continuity, every diffeomorphism is a homeomorphism (but not the other way around!).
(11) A topological manifold may or may not have any differentiable structure. If it does have one, it is sometimes unique, for example if the dimension is ≤ 3, but often it is not unique. There exist topological manifolds which have uncountably many distinct differentiable structures, for example $\mathbb{R}^4$.

20 October 2015

(12) Dimensions of manifolds and smooth invariance of domain.
(13) Examples of differentiable manifolds and their dimensions: Euclidean spaces $\mathbb{R}^n$, spheres S^n, tori T^n, real projective spaces $\mathbb{R}P^n$ and complex projective spaces $\mathbb{C}P^n$. An open subset of a manifold is a manifold (of the same dimension); products of manifolds are manifolds (and the dimensions add up).

(14) If we retain from a smooth atlas for a manifold M only the knowledge of the images of charts, together with the identifications that are to be performed according to the transition maps, then we can reconstruct M; see [1, Sections 3.1 and 3.2].

22 October 2015

(15) The **tangent bundle** TM of a differentiable manifold M of dimension n is itself a differentiable manifold of dimension $2n$.

(16) The natural projection $\pi\colon TM \longrightarrow M$ is differentiable. The preimage $T_xM = \pi^{-1}(x)$ of any point $x \in M$ has a well-defined structure as an n-dimensional real vector space. We call this the **tangent space** of M at x.

(17) For any differentiable map $f\colon M \longrightarrow N$, we define the **derivative** $Df\colon TM \longrightarrow TN$. This restricts to every tangent space T_xM as a linear map $D_xf\colon T_xM \longrightarrow T_{f(x)}N$, called the derivative of f at $x \in M$.

27 October 2015

(18) **Immersions** and **submersions**; examples.

(19) **Submanifolds** and **embeddings**. Not every injective immersion is an embedding.

(20) Applications of the inverse function theorem to the normal form of a submersion. Preimages of points under submersions are submanifolds.

29 October 2015

(21) Every manifold M is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering $\{U_i\}_{i\in I}$ of M, there is an atlas $\{(V_k,\varphi_k)\}$ such that the covering by the V_k is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball B_3 of radius 3 for all k and the open sets $W_k = \varphi_k^{-1}(B_1)$ cover M.

Proof. We prove first that there is a sequence G_i, $i = 1, 2, \dots$ of open sets with compact closures, such that the G_i cover M and satisfy

$$\overline{G_i} \subset G_{i+1}$$

for all i. To this end let A_i, $i = 1, 2, \dots$ be a countable basis of the topology consisting of open sets with compact closures. Set $G_1 = A_1$. Suppose inductively that we have defined

$$G_k = A_1 \cup \ldots \cup A_{j_k}\ .$$

Then let j_{k+1} be the smallest integer greater than j_k with the property that

$$\overline{G_k} \subset A_1 \cup \ldots \cup A_{j_{k+1}}\ ,$$

and define

$$G_{k+1} = A_1 \cup \ldots \cup A_{j_{k+1}}\ .$$

This defines the sequence G_k as desired.

Let $\{U_i\}_{i\in I}$ be an arbitrary open covering of M. For every $x \in M$ we can find a chart (V_x,φ_x) at x with V_x contained in one of the U_i and such that $\varphi_x(V_x) = B_3$. Let $W_x = \varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_k} \setminus G_{k-1}$ by finitely many such W_{x_j} such that at the same time the corresponding V_{x_j} are contained in the open set $G_{k+1} \setminus \overline{G_{k-2}}$. Taking all these V_{x_j} as i ranges over the positive integers we obtain the desired atlas. $\square$

(22) We construct smooth bump functions on $\mathbb{R}^n$ and transfer them to differentiable manifolds via charts. This allows us to construct various kinds of differentiable functions with special properties.

(23) Every open covering of a differentiable manifold admits a subordinate differentiable **partition of unity**. This follows from paracompactness and the existence of smooth bump functions.

3 November 2015

(24) The **Whitney embedding theorem**: every n-dimensional differentiable manifold embeds differentiably in $\mathbb{R}^{2n+1}$. This is true for all (paracompact) manifolds, but we gave the proof only for compact ones.

(25) A particular consequence of the embedding of manifolds in Euclidean spaces is the existence of **Riemannian metrics**, that is, fiberwise positive definite scalar products on the tangent bundle. One just restricts the scalar product of the Euclidean space to the tangent spaces.

5 November 2015

(26) Differentiable **vector bundles** over manifolds; see [1] Section 3.3. Local vs. global triviality; isomorphisms of bundles.

(27) Examples of vector bundles: product bundles, the tangent bundle of a differentiable manifold, the Möbius band.

(28) Sections of vector bundles, and the characterization of triviality of bundles through the existence of sufficiently many sections that are pointwise linearly independent.

10 November 2015

(29) Cocycles of transition maps for systems of local trivializations for vector bundles. Reconstructing a vector bundle from a cocycle. See [1, Section 3.4].

(30) Isomorphism classes of rank k vector bundles correspond to certain equivalence classes of cocycles, which we denote by $H^1(M; GL_k(\mathbb{R}))$.

(31) Some linear algebra of vector bundles: dualization, Whitney sum, subbundles.

12 November 2015

(32) For a subgroup $G \subset GL_k(\mathbb{R})$, a G**-structure** on a rank k vector bundle is a system of local trivializations whose associated cocycle takes values in G. (This is called a G-*reduction* in [1].) Isomorphism classes of G-structures correspond to equivalence classes of cocycles with values in G, denoted $H^1(M; G)$.

(33) The forgetful map $H^1(M; G) \longrightarrow H^1(M; GL_k(\mathbb{R}))$, which sends a G-structure to the isomorphism class of the underlying bundle, is neither injective nor surjective in general. The failure of surjectivity means that there are bundles not admitting a G-structure for a given G, and the failure of injectivity means that certain bundles may have several different G-structures.

(34) Examples of G-structures:
  (1) For $G = \{e\}$ the trivial group, G-structures are global trivializations.
  (2) For $G = GL_k^+(\mathbb{R})$, the group of orientation-preserving isomorphisms of $\mathbb{R}^k$, G-structures are **orientations**. A vector bundle is called orientable, if it admits an orientation.
  (3) If $k = 2m$ is even, we identify $\mathbb{R}^{2m} = \mathbb{C}^m$, and consider $G = GL_m(\mathbb{C})$ as a subgroup of $GL_{2m}(\mathbb{R})$. In this case a G-structure is a complex structure. A (real) vector bundle with a complex structure is, in an obvious way, a complex vector bundle.
  (4) For $G = O(k)$ the orthogonal group, a G-structure is a smooth positive-definite **metric** on the fibers.

(35) If $H \subset G$ is a subgroup, then any H-structure gives rise to a G-structure. For example, every complex structure gives rise to an orientation.

(36) Using partitions of unity we proved that every vector bundle has a metric, i.e. it admits an $O(k)$-structure. This is the only case in the above list of examples where $H^1(M; G) \longrightarrow H^1(M; GL_k(\mathbb{R}))$ is surjective for every M.

(37) If $F \subset E$ is a subbundle, then choosing a metric on E and taking fiber-wise orthogonal complements defines another subbundle $F^\perp$, with the property that E is isomorphic to $F \oplus F^\perp$.

(38) The proof of existence of a metric on every vector bundle relies on positive definiteness to ensure that convex combinations of metrics are again metrics. Indeed, the existence is false for indefinite metrics. For example, by a light cone argument, $TS^2 \longrightarrow S^2$ does not admit any metric of signature $(1, 1)$.

17 November 2015

(39) An **infinitesimal G-structure** on a smooth manifold M is a G-structure on the vector bundle $\pi \colon TM \longrightarrow M$. Such a structure is said to be **integrable** if it arises from the derivatives of the transition maps for a suitable atlas for M.

(40) We discussed infinitesimal G-structures and their integrability in the following examples:
  (1) For $G = \{e\}$ the trivial group, an infinitesimal G-structure on M is a trivialization of the tangent bundle. This can only be integrable if M has an atlas whose transition maps are translations. The easiest example of a manifold with trivial tangent bundle but no integrable trivialization is $S^2 \times S^1$.
  (2) For $G = GL_n^+(\mathbb{R})$, the group of orientation-preserving isomorphisms of $\mathbb{R}^n$, infinitesimal G-structures are **orientations** of the vector bundle $\pi \colon TM \longrightarrow M$. We proved that these structures are always integrable. We define M to be orientable if it admits an infinitesimal $GL_n^+(\mathbb{R})$-structure. A choice of orientation is an isomorphism class of such structures.
  (3) If $n = 2m$ is even, we identify $\mathbb{R}^{2m} = \mathbb{C}^m$, and consider $G = GL_m(\mathbb{C})$ as a subgroup of $GL_{2m}(\mathbb{R})$. In this case an infinitesimal G-structure on M is a complex structure on the tangent bundle. (This is sometimes called an almost complex structure on M.) It is integrable if an only if it arises from at atlas for M whose transition maps have $\mathbb{C}$-linear derivatives, i.e. they are holomorphic maps. Such an atlas makes M into a **complex manifold**.

(4) For $G = O(n)$ the orthogonal group, an infinitesimal G-structure on M is a smooth positive-definite **metric** on the fibers of the tangent bundle. This is called a **Riemannian metric**. It is integrable only if the curvature tensor of the metric vanishes identically.

(5) For $G = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mid A \in GL_l(\mathbb{R}), C \in GL_{n-l}(\mathbb{R}) \right\}$, the subgroup of $GL_n(\mathbb{R})$ preserving the first factor in the product decomposition $\mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^{n-l}$, an infinitesimal G-structure on M is a subbundle $F \subset TM$ of rank l. Such a structure is integrable if and only if M has a **foliation** by l-dimensional leaves tangent to F. Later in this course we will prove the so-called Frobenius theorem, which gives a necessary and sufficient condition for the integrability of a subbundle.

19 November 2015

(41) **Pullbacks** and homomorphisms of vector bundles.

(42) **Global flows** on manifolds, and the **vector fields** obtained by differentiation.

24 November 2015

(43) **Local flows** obtained by locally integrating a vector fields. The correspondence between vector fields and equivalence classes of local flows.

(44) **Completeness** of vector fields. Every vector field with compact support is complete.

(45) Vector fields X act as derivations on smooth functions f by the **Lie derivative** L_X.

26 November 2015

(46) The Lie derivative gives an isomorphism of vector spaces
$$\begin{aligned} \mathcal{X}(M) &\longrightarrow \mathrm{Der}(C^\infty(M)) \\ X &\longmapsto L_X \ . \end{aligned}$$

(47) Since for any two vector fields X and Y, the map $L_X L_Y - L_Y L_X$ is a derivation, by the above isomorphism there is a unique vector field $[X, Y]$ such that $L_{[X,Y]} = L_X L_Y - L_Y L_X$. This is called the **commutator** of X and Y.

(48) The commutator of vector fields satisfies three important properties: it is $\mathbb{R}$-bilinear, it is skew-symmetric, and the following Jacobi identity holds
$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 \quad \forall X, Y, Z \in \mathcal{X}(M) \ .$$
This means that the bracket makes the vector space $\mathcal{X}(M)$ into a **Lie algebra**.

(49) A **Lie group** G is a smooth manifold with a group structure such that multiplication and taking inverses are smooth maps. Basic examples are $\mathbb{R}^n$ with addition of vectors giving the group structure, and $GL_k(\mathbb{R})$ with composition, or matrix multiplication. Any subgroup of $GL_k(\mathbb{R})$ that is a submanifold is also a Lie group.

(50) If $M = G$ is a Lie group, then the left-invariant vector fields on G form a finite-dimensional sub-Lie algebra $\mathfrak{g} \subset \mathcal{X}(G)$. Its dimension, as a vector space, agrees with the dimension of G as a manifold.

1 December 2015

(51) We can also define a Lie derivative acting on vector fields, rather than functions. It turns out that $L_X Y = [X, Y]$.
(52) Two vector fields commute if and only if their local flows commute.
(53) We defined **integral manifolds** for subbundles $E \subset TM$. Such a subbundle is **integrable** if there is an integral manifold through every point. This is equivalent to the integrability of the infinitesimal G-structure defined by E.

3 December 2015

(54) We proved the following:

Theorem 1. *Let $X_1, \ldots, X_k \in \mathcal{X}(M)$ be such that $[X_i, X_j] \equiv 0$ for all i and j. If $p \in M$ is such that $X_1(p), \ldots, X_k(p)$ are linearly independent in T_pM, then there is a chart (U, φ) for M with $p \in M$ such that $D\varphi(X_i|_U) = \frac{\partial}{\partial x_i}$ for all $1 \leq i \leq k$.*

(55) A k**-flow** on a smooth manifold M is a smooth group action of the additive group of $\mathbb{R}^k$ on M. Equivalently, it consists of k global flows, or k complete vector fields, that commute pairwise. A k-flow is non-singular if the corresponding k vector fields are everywhere pointwise linearly independent.
(56) An application of the Theorem above shows that if an n-dimensional connected smooth manifold M admits a non-singular n-flow, then it is diffeomorphic to a product $T^\ell \times \mathbb{R}^{n-\ell}$ for some ℓ. Therefore, if such an M is compact, it is a torus.
(57) Recall that an infinitesimal G-structure on M for G the trivial group is a trivialization of the tangent bundle of M. If M is compact and connected, but not a torus, then the previous item shows that the G-structure cannot be integrable. This applies for example to $M = S^3$ and to $M = S^1 \times S^2$.
(58) Another application of the above Theorem is:

Theorem 2 (Frobenius Theorem)**.** *Let $E \subset TM$ be a subbundle of rank k. The following are equivalent:*
(a) *E is integrable,*
(b) *$\Gamma(E)$ is closed under $[\ ,\ ]$,*
(c) *there is a covering of M by the domains of charts (U, φ) with the property that $\frac{\partial}{\partial x_i} \in D\varphi(E)$ for all $1 \leq i \leq k$.*

If one, and therefore all, of these properties hold, then E is the tangent bundle along the leaves of a k-dimensional foliation on M, whose leaves are the integral manifolds of E. The charts in the last item are the foliated charts for the foliation.

8 December 2015

(59) Some more details on the proof of the Frobenius Theorem.
(60) A **differential form** of degree k on a smooth manifold M is a map

$$\omega \colon \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \longrightarrow C^\infty(M)$$

of $C^\infty(M)$-modules, in other words, it is function-linear in all k arguments. In addition, it is required to satisfy the following condition:

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = sign(\sigma)\omega(X_1, \ldots, X_k) \tag{1}$$

for all permutations $\sigma \in S_k$.

(61) We have the following:

Lemma 3. *If ω is a differential form, then the value of the function $\omega(X_1, \ldots, X_k)$ at a point $p \in M$ depends on the vector fields X_i only through their values $X_i(p)$ at the point p.*

This means that ω has a value ω_p at p, which is a k-multilinear map

$$\omega_p \colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$$

defined on $(X_1(p), \ldots, X_k(p))$ by extending the $X_i(p)$ to global vector fields, evaluating ω on these vector fields, and then evaluating the resulting function at p. (This multilinear map of course inherits property (1).)

(62) We build a universal model for multilinear maps, first for vector spaces (like T_pM), and then for vector bundles (like TM). This will allow us to interpret differential forms as sections of suitable vector bundles, so that ω_p will be simply the value of the section ω at p. The universal model for bilinear maps on $V \times W$ is given by the **tensor product** $V \times W \longrightarrow V \otimes W$. (See [1] Section 7.1.). We proved that the tensor product of vector spaces is uniquely characterized by its universal property.

10 December 2015

(63) Existence of the tensor product.

(64) Iterating the construction of the tensor product we obtain tensor products of k vector spaces which have the universal property for k-linear maps. The **tensor algebra** of a vector space V is the direct sum of the tensor products $T^k(V)$ of k copies of V, for $k = 0, 1, 2, \ldots$ endowed with the natural mutiplication given by the tensor product. Here $T^0(V)$ is just the ground field, and $T^1(V)$ is V itself. The tensor algebra is a graded associative algebra. (See [1, Section 7.1].)

(65) For skew-symmetric multilinear maps there is a universal model $V \times \ldots \times V \longrightarrow \Lambda^k V$ obtained as the quotient of $T^k(V)$ by the intersection of $T^k(V)$ with the alternating ideal in the tensor algebra.

(66) The **exterior algebra** of a vector space V over a field of characteristic $\neq 2$ is the direct sum

$$\Lambda(V) = \bigoplus_{k=0}^{\infty} \Lambda^k V$$

(See [1, Section 7.2].)

(67) We compute the dimensions of tensor end exterior products as follows:

$$\dim(V \otimes W) = \dim V \cdot \dim W \ ,$$

$$\dim \Lambda^k V = \binom{\dim V}{k} \ .$$

This shows in particular that the exterior algebra, unlike the tensor algebra, is non-trivial in only finitely many degrees.

(68) **Induced maps** on the tensor algebra and on the exterior algebra. The following lemma will be important:

Lemma 4. *If V is a vector space of dimension n and $f\colon V \longrightarrow V$ is a linear map, then the induced map $\lambda(f)\colon \Lambda^n(V) \longrightarrow \Lambda^n(V)$ is multiplication by the determinant $\det(f)$.*

The proof is left as a homework exercise.

15 December 2015

(69) Multilinear algebra constructions applied to vector bundles. See [1, Section 7.4].
(70) **Differential forms** as sections of exterior powers of the cotangent bundle.
(71) Orientability and orientations on vector bundles via their maximal exterior powers.
(72) Orientability and orientations on manifolds via the co-/tangent bundle and its maximal exterior power. Orientability is equivalent to the existence of a volume form.

17 December 2015

(73) **Exterior derivatives**; existence and uniqueness, and explicit formulas in small degrees.
(74) **Pullback** of differential forms. The pullback commutes with the exterior derivative.

22 December 2015

(75) Contractions and Lie derivatives of differential forms.
(76) **Cartan's formula** $L_X = i_X \circ d + d \circ i_X$.
(77) **Manifolds with boundary**.

7 January 2016

(78) Tangent spaces and tangent bundles for manifolds with boundary.
(79) Orientations for manifolds with boundary and the induced orientations on the boundary.
(80) The **integral** of n-forms with compact support on oriented n-manifolds. (See [1] Section 8.2.)

12 January 2016

(81) The integral is well-defined.
(82) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \ .$$

(See [1] Section 8.2.)

14 January 2016

(83) **Closed** and **exact** k-forms; the de Rham complex and its cohomology, called the **de Rham cohomology** $H^k_{dR}(M)$ of a differentiable manifold M.
(84) The wedge product of forms induces a well-defined multiplication on de Rham cohomology, making the total de Rham cohomology of a manifold into a graded algebra.

(85) The forms with compact support form a subcomplex of the de Rham complex. Its cohomology is called the (de Rham) cohomology with compact support and denoted $H^k_c(M)$. For compact manifolds this is of course the same as the ordinary de Rham cohomology defined above.

(86) Calculations of de Rham cohomology (with or without compact supports) for $\mathbb{R}$.

(87) For any oriented n-dimensional manifold M without boundary, the integral gives a well-defined surjective linear map:

$$\int_M \colon H^n_c(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega \ .$$

19 January 2016

(88) A differentiable map $f \colon M \longrightarrow N$ induces a map on de Rham cohomology

$$f^* \colon H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

defined by pulling back closed forms. (Recall that on forms the pullback commutes with exterior differentiation.)

(89) The **Poincaré lemma**: If $i_{t_0} \colon M \longrightarrow M \times \mathbb{R}$ is the inclusion of M as $M \times \{t_0\}$ and $\pi \colon M \times \mathbb{R} \longrightarrow M$ is the projection, then $i^*_{t_0}$ and π^* are inverses of each other on cohomology. Thus M and $M \times \mathbb{R}$ have isomorphic de Rham cohomology.

(90) Consequences of the Poincaré lemma: smoothly homotopic maps induce the same homomorphism on de Rham cohomology, smoothly homotopy equivalent smooth manifolds have the same de Rham cohomology, in particular contractible manifolds have the de Rham cohomology of a point. This means that all closed forms are locally exact.

(91) As consequences of the Poincaré lemma we have in particular a complete calculation of the de Rham cohomology of $\mathbb{R}^n$ by induction on n.

21 January 2016

(92) Some more details for the proof of the Poincaré lemma.

(93) By induction on n we find the cohomology of $\mathbb{R}^n$ with compact supports, and, at the same time, the de Rham cohomology of S^n. In degree n these are both one-dimensional, with the isomorphism to $\mathbb{R}$ given by integration.

(94) For any connected oriented n-dimensional manifold M without boundary, we have $H^n_c(M) = \mathbb{R}$, with the isomorphism again given by integration.

26 January 2016

(95) A smooth manifold M is called **closed** if it is compact without boundary. For now we only consider connected oriented manifolds, so often closed manifolds are implicitly assumed to be connected and oriented.

(96) Let $f\colon M \longrightarrow N$ be a smooth map between closed n-dimensional manifolds. (As said above, M and N are assumed to be connected and oriented.) Then $f^*\colon H^n_{dR}(N) \longrightarrow H^n_{dR}(M)$ is a linear map between one-dimensional real vector spaces. We identify both these vector spaces with $\mathbb{R}$ using the isomorphism given by integration. Then $f^*\colon \mathbb{R} \longrightarrow \mathbb{R}$ is multiplication by a real number λ, which we call the **degree** of f, and denote by $\deg(f)$.

(97) Here are the most basic properties of the degree:

(0) If f and g are two (smoothly) homotopic maps, then $\deg(f) = \deg(g)$, since in this case $f^* = g^*$.

(1) If $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow L$, then $\deg(g \circ f) = \deg(f) \cdot \deg(g)$, since in this case $(g \circ f)^* = f^* \circ g^*$.

(2) If there is a compact oriented manifold W with boundary $\partial W = M$, and an extension of $f\colon M \longrightarrow N$ to a smooth map $F\colon W \longrightarrow N$, then $\deg(f) = 0$.

(3) If f is a diffeomorphism, then $\deg(f) = \pm 1$, according to whether f is orientation-preserving, or not.

(4) The degree is always an integer, in fact, it is the algebraic number of preimages of a regular value of f, where preimages are counted with signs, according to whether the derivative of f at the point is orientation-preserving, or not. In particular, if f is not surjective, then $\deg(f) = 0$.

(98) As a first application of the degree, we proved the following statement: If W is a compact oriented smooth manifold with boundary $\partial W = M$, then there is no smooth map $r\colon W \longrightarrow M$ with the property $r|_{\partial W} = Id_M$. (Such an r would be called a retraction of W onto its boundary.) A direct derivation of this statement from Stokes's theorem is possible, without using the degree. However, once we have the above properties of the degree, the argument is purely formal. First of all, we may assume that W is connected. If r were to exist, then M would be the continuous image of a connected manifold, and therefore connected. Now M is closed, connected and oriented, and by property (3) above, we have $\deg(Id_M) \neq 0$. However, if r were to exist, we would conclude from property (2) that $\deg(Id_M) = 0$. This contradiction completes the proof.

(99) From the fact that the closed ball $\overline{B_1(0)} \subset \mathbb{R}^n$ does not retract onto its boundary S^{n-1} we deduced the **Brouwer fixed point theorem**: every smooth map $f\colon \overline{B_1(0)} \longrightarrow \overline{B_1(0)}$ has a fixed point.

(100) A somewhat deeper application of the notion of degree is the proof of the **hairy ball theorem**: The sphere S^n admits a vector field without zeroes if and only if n is odd. A vector field without zeroes allows us to produce a homotopy between the identity and the antipodal map of S^n. Now the degree of the antipodal map on S^n is $(-1)^{n+1}$ by applying property (3) above, so if n is even we get a contradiction with property (0).

References

1. L. Conlon, *Differentiable Manifolds — A First Course*, Birkhäuser Verlag 1993.
2. S. Morita, *Geometry of Differential Forms*, Amer. Math. Soc. 2001.
3. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Springer Verlag 1983.