Differential geometry / Differenzierbare Mannigfaltigkeiten
WS 2013/14

D. Kotschick

15 October 2013

(1) Topological spaces; Hausdorffness; continuity.
(2) Metric spaces, and their induced topology. This topology is always Hausdorff.
(3) Euclidean space $\mathbb{R}^n$ with its metric topology. By considering balls with rational radii centered at points with rational coordinates, one finds a countable collection of open sets such that all open sets are suitable unions of these.

16 October 2013

(4) Bases for topologies. Remark that by the previous item, Euclidean space has a countable bases for its topology.
(5) **Topological manifolds** are locally Euclidean spaces that are Hausdorff and have a countable basis for their topology.
(6) A **differentiable manifold** is a topological manifold together with an atlas whose transition maps are differentiable. Such an atlas is called a differentiable or smooth atlas.
(7) We define differentiability for maps between differentiable manifolds using charts. A special case concerns real-valued functions.
(8) A **differentiable structure** is an equivalence class of atlases, equivalently a maximal atlas.
(9) If we retain from a smooth atlas for a manifold M only the knowledge of the images of charts, together with the identifications that are to be performed according to the transition maps, then we can reconstruct M; see [1, Sections 3.1 and 3.2].
(10) Every maximal C^r atlas with $r \geq 1$ contains a C^∞ atlas. Because of this fact (which we do not prove), we will restrict ourselves to C^∞ manifolds throughout. So the words differentiable or smooth will usually mean C^∞.
(11) A topological manifold may or may not have any differentiable structure. If it does have one, it is sometimes unique, for example if the dimension is ≤ 3, but often it is not unique. There exist topological manifolds which have uncountably many distinct differentiable structures, for example $\mathbb{R}^4$.

22 October 2013

(12) Dimensions of manifolds and smooth invariance of domain.
(13) Examples of differentiable manifolds and their dimensions: Euclidean spaces $\mathbb{R}^n$, spheres S^n, tori T^n, $GL(n, \mathbb{R})$,... An open subset of a manifold is a manifold (of the same dimension); products of manifolds are manifolds (and the dimensions add up).
(14) The **tangent bundle** TM of a differentiable manifold M of dimension n is itself a (topological) manifold of dimension $2n$.

23 October 2013

(15) The tangent bundle of a differentiable manifold is in fact a differentiable manifold, since the natural atlas is differentiable.

(16) The natural projection $\pi\colon TM \longrightarrow M$ is differentiable. The preimage $T_xM = \pi^{-1}(x)$ of any point $x \in M$ has a well-defined structure as an n-dimensional real vector space. We call this the **tangent space** of M at x.

(17) For any differentiable map $f\colon M \longrightarrow N$, we define the **derivative** $Df\colon TM \longrightarrow TN$. This restricts to every tangent space T_xM as a linear map $D_xf\colon T_xM \longrightarrow T_{f(x)}N$, called the derivative of f at $x \in M$.

29 October 2013

(18) Differentiable **vector bundles** over manifolds; see [1] Section 3.3. Local vs. global triviality; isomorphisms of bundles.

(19) Examples of vector bundles: product bundles, the tangent bundle of a differentiable manifold, the Möbius band.

(20) Sections of vector bundles, and the characterization of triviality of bundles through the existence of sufficiently many sections that are pointwise linearly independent.

30 October 2013

(21) Cocycles of transition maps for systems of local trivializations for vector bundles. Reconstructing a vector bundle from a cocycle. See [1, Section 3.4].

(22) Some linear algebra of vector bundles: dualization, Whitney sum, subbundles.

(23) For a subgroup $G \subset GL_k(\mathbb{R})$, a G-structure on a rank k vector bundle is a system of local trivializations whose associated cocycle takes values in G. (This is called a *G-reduction* in [1].)

(24) Orientations of vector bundles as $GL_k^+(\mathbb{R})$-structures.

5 November 2013

(25) Every manifold M is paracompact, meaning that every open cover has an open locally finite refinement. We prove the following more precise statement. Given an open covering $\{U_i\}_{i\in I}$ of M, there is an atlas $\{(V_k, \varphi_k)\}$ such that the covering by the V_k is a locally finite refinement of the given covering, and such that $\varphi_k(V_k)$ is an open ball B_3 of radius 3 for all k and the open sets $W_k = \varphi_k^{-1}(B_1)$ cover M.

Proof. We prove first that there is a sequence G_i, $i = 1, 2, \ldots$ of open sets with compact closures, such that the G_i cover M and satisfy

$$\overline{G_i} \subset G_{i+1}$$

for all i. To this end let A_i, $i = 1, 2, \ldots$ be a countable basis of the topology consisting of open sets with compact closures. Set $G_1 = A_1$. Suppose inductively that we have defined

$$G_k = A_1 \cup \ldots \cup A_{j_k} .$$

Then let j_{k+1} be the smallest integer greater than j_k with the property that

$$\overline{G_k} \subset A_1 \cup \ldots \cup A_{j_{k+1}} ,$$

and define

$$G_{k+1} = A_1 \cup \ldots \cup A_{j_{k+1}} .$$

This defines the sequence G_k as desired.

Let $\{U_i\}_{i\in I}$ be an arbitrary open covering of M. For every $x \in M$ we can find a chart (V_x, φ_x) at x with V_x contained in one of the U_i and such that $\varphi_x(V_x) = B_3$. Let $W_x = \varphi_x^{-1}(B_1)$. We can cover each set $\overline{G_k} \setminus G_{k-1}$ by finitely many such W_{x_j} such that at the same time the corresponding V_{x_j} are contained in the open set $G_{k+1} \setminus \overline{G_{k-2}}$. Taking all these V_{x_j} as i ranges over the positive integers we obtain the desired atlas. □

(26) We construct smooth bump functions on $\mathbb{R}^n$ and transfer them to differentiable manifolds via charts. This allows us to construct various kinds of differentiable functions with special properties.

(27) Every open covering of a differentiable manifold admits a subordinate differentiable **partition of unity**. This follows from paracompactness and the existence of smooth bump functions.

6 November 2013

(28) Let $\pi\colon E \longrightarrow M$ be a vector bundle of rank $k > 0$ over a base manifold of positive dimension. We have the following applications of the existence of smooth bump functions and of partitions of unity:
- the evaluation map
$$\begin{aligned} ev\colon M \times \Gamma(E) &\longrightarrow E \\ (p, s) &\longmapsto s(p) \end{aligned}$$
is surjective,
- the space of sections $\Gamma(E)$ is infinite-dimensional,
- E admits a fibre-wise positive definite scalar product depending smoothly on the fiber; this is called a **metric** on E, and the existence of a metric is equivalent to the existence of an $O(k)$-structure,
- every subbundle $F \subset E$ admits a complementary subbundle given by the orthogonal complement $F^\perp$ with respect to a metric, so that E is isomorphic to $F \oplus F^\perp$.

(29) The proof of existence of a metric relies on positive definiteness to ensure that convex combinations of metrics are again metrics. Indeed, the existence is false for indefinite metrics. For example, by a light cone argument, $TS^2 \longrightarrow S^2$ does not admit any metric of signature $(1,1)$.

12 November 2013

(30) **Immersions**, **embeddings** (= proper, injective, immersions), **submanifolds**.

(31) The **Whitney embedding theorem**: every n-dimensional differentiable manifold embeds differentiably in $\mathbb{R}^{2n+1}$. This is true for all (paracompact) manifolds, but we gave the proof only for compact ones.

13 November 2013

(32) **Global flows** on manifolds, and the vector fields obtained by differentiation.

(33) **Local flows** obtained by locally integrating a vector fields. The correspondence between vector fields and equivalence classes of local flows.

(34) **Completeness** of vector fields. Every vector field with compact support is complete.

19 November 2013

(35) Vector fields X act as derivations on smooth functions f by the **Lie derivative** L_X. This gives an isomorphism of vector spaces

$$\begin{aligned} \mathcal{X}(M) &\longrightarrow \mathrm{Der}(C^\infty(M)) \\ X &\longmapsto L_X \ . \end{aligned}$$

(36) Since for any two vector fields X and Y, the map $L_X L_Y - L_Y L_X$ is a derivation, by the above isomorphism there is a unique vector field $[X, Y]$ such that $L_{[X,Y]} = L_X L_Y - L_Y L_X$. This is called the **commutator** of X and Y.

20 November 2013

(37) The commutator of vector fields satisfies the following three properties: it is $\mathbb{R}$-bilinear, it is skew-symmetric, and the Jacobi identity holds:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 \quad \forall X, Y, Z \in \mathcal{X}(M) \ .$$

This means that the bracket makes the vector space $\mathcal{X}(M)$ into a **Lie algebra**.

(38) If $M = G$ is a Lie group, then the left-invariant vector fields on G form a finite-dimensional sub-Lie algebra $\mathfrak{g} \subset \mathcal{X}(G)$. Its dimension, as a vector space, agrees with the dimension of G, as a manifold.

(39) We can also define a Lie derivative acting on vector fields, rather than functions. It turns out that $L_X Y = [X, Y]$.

26 November 2013

(40) Two vector fields commute if and only if their local flows commute.

(41) A **differential form** of degree k on a smooth manifold M is a map

$$\omega \colon \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \longrightarrow C^\infty(M)$$

of $C^\infty(M)$-modules, in other words, it is function-linear in all k arguments. In addition, it is required to satisfy the following condition:

$$\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}) = sign(\sigma)\omega(X_1, \ldots, X_k) \tag{1}$$

for all permutations $\sigma \in S_k$.

(42) We have the following:

Lemma 1. *If ω is a differential form, then the value of the function $\omega(X_1, \ldots, X_k)$ at a point $p \in M$ depends on the vector fields X_i only through their values $X_i(p)$ at the point p.*

This means that ω has a value ω_p at p, which is a k-multilinear map

$$\omega_p \colon T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$$

defined on $(X_1(p), \ldots, X_k(p))$ by extending the $X_i(p)$ to global vector fields, evaluating ω on these vector fields, and then evaluating the resulting function at p. (This multilinear map of course inherits property (1).)

(43) We build a universal model for multilinear maps, first for vector spaces (like T_pM), and then for vector bundles (like TM). This will allow us to interpret differential forms as sections of suitable vector bundles, so that ω_p will be simply the value of the section ω at p. The universal model for bilinear maps on $V \times W$ is given by the **tensor product** $V \times W \longrightarrow V \otimes W$. (See [1] Section 7.1.).

27 November 2013

(44) Iterating the construction of the tensor product we obtain tensor products of k vector spaces which have the universal property for k-linear maps. The **tensor algebra** of a vector space V is the direct sum of the tensor products $T^k(V)$ of k copies of V, for $k = 0, 1, 2, \ldots$ endowed with the natural mutiplication given by the tensor product. Here $T^0(V)$ is just the ground field, and $T^1(V)$ is V itself. The tensor algebra is a graded associative algebra. (See [1, Section 7.1].)

(45) For skew-symmetric multilinear maps there is a universal model $V \times \ldots \times V \longrightarrow \Lambda^k V$ obtained as the quotient of $T^k(V)$ by the intersection of $T^k(V)$ with the alternating ideal in the tensor algebra.

(46) The **exterior algebra** of a vector space V over a field of characteristic $\neq 2$ is the direct sum

$$\Lambda(V) = \bigoplus_{k=0}^{\infty} \Lambda^k V$$

(See [1, Section 7.2].)

(47) We compute the dimensions of tensor end exterior products as follows:

$$\dim(V \otimes W) = \dim V \cdot \dim W \ ,$$

$$\dim \Lambda^k V = \binom{\dim V}{k} \ .$$

This shows in particular that the exterior algebra, unlike the tensor algebra, is non-trivial in only finitely many degrees.

(48) **Induced maps** on the tensor algebra and on the exterior algebra. The following lemma will be important:

Lemma 2. *If V is a vector space of dimension n and $f \colon V \longrightarrow V$ is a linear map, then the induced map $\Lambda^n(f) \colon \Lambda^n(V) \longrightarrow \Lambda^n(V)$ is multiplication by the determinant $\det(f)$.*

The proof is left as a homework exercise.

3 December 2013

(49) Multilinear algebra constructions applied to vector bundles. See [1, Section 7.4].

(50) **Differential forms** as sections of exterior powers of the cotangent bundle. The wedge product of forms.

(51) **Exterior derivatives**.

4 December 2013

(52) Existence and uniqueness of the exterior derivative.

(53) **Pullback** of differential forms. The pullback commutes with the exterior derivative.

(54) Interpretation of the derivative of a smooth map $f\colon M \longrightarrow N$ as a section of the vector bundle $\mathrm{Hom}(TM, f^*TN) = T^*M \otimes f^*TN$ over M.

10 December 2013

(55) Orientability and orientations on manifolds via the co-/tangent bundle and its maximal exterior power. Orientability is equivalent to the existence of a volume form.
(56) The **integral** of n-forms with compact support on oriented n-manifolds. (See [1] Section 8.2.)
(57) **Manifolds with boundary**; their boundaries and their interiors. Every manifold (in the usual sense) is also a "manifold with boundary", but the boundary happens to be empty.

11 December 2013

(58) Orientations of manifolds with boundary and the induced orientation on the boundary.
(59) **Stokes's Theorem** for oriented manifolds with boundary:

$$\int_M d\omega = \int_{\partial M} \omega \ .$$

(See [1] Section 8.2.)
(60) Easy consequences from Stokes's theorem...

17 December 2013

(61) **Closed** and **exact** k-forms; the de Rham complex and its cohomology, called the **de Rham cohomology** $H^k_{dR}(M)$ of a differentiable manifold M.
(62) The wedge product of forms induces a well-defined multiplication on de Rham cohomology, making the total de Rham cohomology of a manifold into a graded algebra.
(63) The forms with compact support form a subcomplex of the de Rham complex. Its cohomology is called the (de Rham) cohomology with compact support and denoted $H^k_c(M)$. For compact manifolds this is of course the same as the ordinary de Rham cohomology defined above.
(64) Calculations of de Rham cohomology (with or without compact supports) for $\mathbb{R}$.
(65) For any oriented n-dimensional manifold M without boundary, the integral gives a well-defined surjective linear map:

$$\int_M \colon H^n_c(M) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_M \omega \ .$$

(66) The **Poincaré lemma**: If $i_{t_0}\colon M \longrightarrow M \times \mathbb{R}$ is the inclusion of M as $M \times \{t_0\}$ and $\pi\colon M \times \mathbb{R} \longrightarrow M$ is the projection, then $i^*_{t_0}$ and π^* are inverses of each other on cohomology. Thus M and $M \times \mathbb{R}$ have isomorphic de Rham cohomology.
(67) More generally, any differentiable map $f\colon M \longrightarrow N$ induces a map on de Rham cohomology

$$f^*\colon H^k_{dR}(N) \longrightarrow H^k_{dR}(M)$$

defined by pulling back closed forms. (Recall that on forms the pullback commutes with exterior differentiation.)

(68) We reduced the proof of the Poincaré lemma to the construction of a certain chain homotopy.

18 December 2013

(69) We produced the chain homotopy required in the proof of the Poincaré lemma.
(70) Consequences of the Poincaré lemma: smoothly homotopic maps induce the same homomorphism on de Rham cohomology, smoothly homotopy equivalent smooth manifolds have the same de Rham cohomology, in particular contractible manifolds have the de Rham cohomology of a point. This means that all closed forms are locally exact.
(71) As consequences of the Poincaré lemma we have in particular a complete calculation of the de Rham cohomology of $\mathbb{R}^n$ by induction on n.
(72) By induction on n we find the cohomology of $\mathbb{R}^n$ with compact supports, and, at the same time, the de Rham cohomology of S^n. In degree n these are both one-dimensional, with the isomorphism given by integration.

7 January 2014

(73) We now begin the discussion of **connections** and **curvature** on vector bundles.
Let $E \to M$ be a differentiable vector bundle of rank k over a smooth manifold M of dimension n.

Definition 3. A **connection** on E is an $\mathbb{R}$-linear map

$$\nabla\colon \Gamma(E) \longrightarrow \Omega^1(E) \tag{2}$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s) \tag{3}$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Here $\Omega^1(E) = \Gamma(T^*M \otimes E)$ is the space of 1-forms on M with values in E. One can evaluate the 1-form on a vector field X to obtain

$$\nabla_X(s) := \langle \nabla(s), X\rangle \in \Gamma(E)\ . \tag{4}$$

(74) We prove the following fundamental properties of connections:
- A connection ∇ does not increase the support of sections, i. e. if $s \in \Gamma(E)$ vanishes on some open set $U \subset M$, then so does $\nabla(s)$.
- The value of $\nabla(s)$ at a point $p \in B$ depends only on the restriction of s to an orbitrarily small open neighbourhood of p. (In other words, ∇ is a differential operator, and $\nabla(s)(p)$ depends only on the germ of s at p.)
- If ∇_1 and ∇_2 are connections, then so is $t\nabla_1 + (1-t)\nabla_2$ for all $t \in [0,1]$.
- If ∇_1 and ∇_2 are connections, then $\nabla_1 - \nabla_2 \in \Omega^1(End(E)) = \Gamma(T^*B \otimes E^* \otimes E)$.

(75) Using these properties and a partition of unity subordinate to a covering of M by open sets over which the restriction of E is trivial, we prove:

Proposition 4. *Every vector bundle E admits connections. The space of all connections on E is an affine space for the space $\Omega^1(End(E))$ of 1-forms on M with values in $End(E)$.*

(76) Next we extend the differential operator given by a connection ∇ to bundle-valued forms of higher degree.

Lemma 5. *For every connection ∇ on $E \to M$ there is a unique $\mathbb{R}$-linear map*

$$\bar{\nabla}\colon \Omega^l(E) \longrightarrow \Omega^{l+1}(E)$$

which satisfies

$$\bar{\nabla}(\omega \otimes s) = d\omega \otimes s + (-1)^l \omega \wedge \nabla(s) \tag{5}$$

for all $\omega \in \Omega^l(M)$ and $s \in \Gamma(E)$. Moreover, this operator satisfies

$$\bar{\nabla}(f(\omega \otimes s)) = (df \wedge \omega) \otimes s + f\bar{\nabla}(\omega \otimes s) \tag{6}$$

for all smooth functions f on M.

8 January 2014

(77) Consider the operator $\bar{\nabla} \circ \nabla\colon \Omega^0(E) \longrightarrow \Omega^2(E)$ associated with a connection ∇ on E. It turns out that this is linear over $C^\infty(M)$, and is therefore given by an element $F^\nabla \in \Omega^2(End(E))$. This is called the **curvature** of ∇.

(78) A (local) **frame** for E is a set of smooth sections $s_1, \ldots, s_k$ defined over some open set $U \subset M$, whose values are linearly independent at every point $p \in U$.

Thus a set of k local smooth sections $s_1, \ldots, s_k$ is a frame if and only if $s_1(p), \ldots, s_k(p)$ is a basis of $E_p = \pi^{-1}(p)$ for every $p \in U$. Therefore a frame defined over U defines a trivialization of $E|_U$, and, conversely, every such trivialization

$$\psi\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

defines a local frame by setting $s_i(p) = \psi^{-1}(p, e_i)$, where $e_1, \ldots, e_k$ is the standard basis of $\mathbb{R}^k$.

(79) Fix a local frame $s_1, \ldots, s_k$ for the restriction of E to a trivialising open set in M. This choice determines a connection ∇_0 defined by the requirement $\nabla_0(s_i) = 0$ for all i. Every other connection ∇ differs from ∇_0 by the addition of a 1-form with values in $End(E)$. However, the given trivialization of E induces a trivialization of $End(E)$, and so a 1-form with values in $End(E)$ is nothing but a $k \times k$ matrix of ordinary 1-forms. Thus ∇ can be expressed by the matrix $\omega = (\omega_{ij})$ of 1-forms given by

$$\nabla(s_i) = \sum_{j=1}^{k} \omega_{ij} \otimes s_j \ .$$

(80) From the definition of the curvature we calculate

$$F^\nabla(s_i) = \sum_{j=1}^{k} \Omega_{ij} \otimes s_j$$

with

$$\Omega_{ij} = d\omega_{ij} - \sum_{l=1}^{k} \omega_{il} \wedge \omega_{lj} \ .$$

We can write this briefly as $\Omega = d\omega - \omega \wedge \omega$, where the wedge product on the right-hand-side includes matrix multiplication, and is therefore not necessarily trivial unless $k = 1$.

(81) Similarly we compute $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$. This is the **Bianchi identity**.

(82) A choice of a (local) frame is called a **choice of gauge** in physics terminology. The connection and curvature matrices represent ∇ and F^∇ with respect to this choice. Connections are referred to as gauge fields.

Suppose we have another frame $s_1', \dots, s_k'$ on the same domain of definition as the original frame. Let ω' and Ω' denote the connection and curvature matrices of ∇ with respect to this new frame. If

$$s_i' = \sum_{i=1}^{k} g_{ij} s_j \ ,$$

we find that $\omega' = dg\ g^{-1} + g\omega g^{-1}$.

14 January 2014

(83) Using $\omega' = dg\ g^{-1} + g\omega g^{-1}$, we calculate $\Omega' = g\Omega g^{-1}$, where $g = (g_{ij})$. The change of basis g is called a **gauge transformation**, and these formulae show how connection and curvature matrices behave under gauge transformations. The curvature matrix Ω is more invariant than the connection matrix ω.

(84) Recall that with respect to a frame $s_1, \dots, s_k$ of E a connection ∇ is expressed by a matrix (ω_{ij}) of one-forms. If we choose a chart for the base manifold M with local coordinates $x_1, \dots, x_n$, then in the domain of this chart every one-form can be expressed uniquely as a linear combination of the dx_i. In particular, there are smooth functions ω_{ij}^α on the domain of the chart such that

$$\omega_{ij} = \sum_{\alpha=1}^{n} \omega_{ij}^\alpha dx_\alpha \ . \tag{7}$$

Denoting the vector fields $\frac{\partial}{\partial x_\alpha}$ by ∂_α, we find the following:

$$\nabla_{\partial_\alpha} s_i = \langle \partial_\alpha, \nabla s_i \rangle = \sum_{j=1}^{k} \langle \partial_\alpha, \omega_{ij} \rangle s_j = \sum_{j=1}^{k} \omega_{ij}^\alpha s_j \ .$$

More generally, if

$$s = \sum_{i=1}^{k} f_i s_i \ ,$$

then

$$\nabla_{\partial_\alpha} s = \sum_{j=1}^{k} \left(\frac{\partial f_j}{\partial x_\alpha} + \sum_{i=1}^{k} f_i \omega_{ij}^\alpha \right) s_j \ .$$

Writing A^α for the matrix (ω_{ij}^α) of functions we see that the covariant derivative ∇_{∂_α}, which we abbreviate to ∇_α, has the form $\nabla_\alpha = \partial_\alpha + A^\alpha$.

(85) We can now give a first geometric interpretation of the curvature, or at least of its vanishing. A connection ∇ is called **flat** if $F^\nabla = 0$.

Proposition 6. $[\nabla_\alpha, \nabla_\beta] s_i = \sum_{j=1}^{k} \Omega_{ij}(\partial_\alpha, \partial_\beta) s_j$

Corollary 7. *The connection ∇ is flat if and only if $[\nabla_\alpha, \nabla_\beta] = 0$ for every local coordinate system $x_1, \dots, x_n$ on the base manifold M.*

Thus the curvature quantifies the failure of the commutativity of covariant derivatives.

(86) If $E \to M$ is a vector bundle with a connection ∇, we say that a section $s \in \Gamma(E)$ is **parallel** with respect to ∇ if $\nabla s = 0$. In the special case that ∇ is the connection given by some trivialization, a section is parallel if and only if it is constant in the given trivialization. Thus parallel sections should be thought of as the analogs of constant sections for nontrivial bundles.

(87) We will want to prove the following:

Proposition 8. *Let $\pi\colon E \to M$ be a smooth vector bundle with a connection ∇, and $c\colon [0,1] \to M$ a smooth curve in the base. Then for every $v \in \pi^{-1}(c(0))$ there is a unique smooth curve $\tilde{c}\colon [0,1] \to E$ with $\pi \circ \tilde{c} = c$, $\tilde{c}(0) = v$ and $\nabla_{\dot{c}} s = 0$, where s sends $c(t)$ to $\tilde{c}(t)$. Moreover, the map $v \mapsto \tilde{c}(1)$ defines a linear map of vector spaces $\pi^{-1}(c(0)) \to \pi^{-1}(c(1))$.*

(88) In Proposition 8 the condition $\nabla_{\dot{c}} s = 0$ makes sense although s is not a section over all of M because the covariant derivative is only considered in the direction of c, where s is defined.

The Proposition follows from the existence and uniqueness of the solutions of systems of linear ordinary differential equations with given initial conditions, together with the linear dependence of the solutions on the initial values.

Definition 9. The linear map

$$\begin{aligned} P_t\colon E_{c(0)} &\longrightarrow E_{c(t)} \\ v &\longmapsto \tilde{c}(t) \end{aligned}$$

is called the **parallel transport** along c. It is an isomorphism of vector spaces.

15 January 2014

(89) As a consequence of Proposition 8 we have:

Corollary 10. *Over a curve every vector bundle with connection admits a framing by parallel sections. Over a one-dimensional base every vector bundle with connection admits local trivializations by parallel frames.*

Here the existence of a parallel frame is over the interval parametrizing the curve. Even if the endpoint of the curve agrees with the starting point, the same may not be true for the initial and ending frames. This is why the second statement is only local.

(90) This corollary fails for base spaces which are not one-dimensional, and this leads to geometric interpretations of the curvature. It will turn out that the corollary encodes the fact that on a one-manifold there is no curvature (as every two-form vanishes identically).

(91) We now prove:

Theorem 11. *A vector bundle $E \to M$ with connection ∇ admits local frames consisting of parallel sections if and only if ∇ is flat, i. e. $F^{\nabla} = 0$.*

(92) One of the consequences of this theorem is:

Corollary 12. *A vector bundle $E \to M$ admits a flat connection if and only if it has a system of local trivializations for which all transition maps are constant.*

21 January 2014

(93) Recall that **metrics** on vector bundles are smoothly varying fiberwise positive-definite scalar products. Using partitions of unity we proved that every vector bundle admits a metric.

(94) Once a metric has been chosen, any local frame can be orthonormalized to obtain a smooth local orthonormal frame.

(95) A connection ∇ on a vector bundle $E \to M$ is compatible with a metric $\langle\ ,\ \rangle$ if and only if

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

for all pairs of sections $s_1, s_2 \in \Gamma(E)$. Sometimes a connection compatible with some metric is called a metric connection.

Lemma 13. *A connection ∇ is compatible with a metric $\langle\ ,\ \rangle$ if and only if its connection matrix ω with respect to any local frame that is orthonormal with respect to $\langle\ ,\ \rangle$ is skew-symmetric, i. e. $\omega_{ij} = -\omega_{ji}$ for all i and j. In this case the curvature matrix Ω with respect to a local orthonormal frame is also skew-symmetric: $\Omega_{ij} = -\Omega_{ji}$.*

Finally, metric connections always exist. The following is proved by combining the above lemma with the proof of Theorem 4.

Proposition 14. *Every vector bundle E with a metric $\langle\ ,\ \rangle$ admits compatible connections. The space of all compatible connections is an affine space for the space $\Omega^1(SkewEnd(E))$ of 1-forms with values in the endomorphisms of E which are skew-symmetric with respect to $\langle\ ,\ \rangle$.*

Here an endomorphism A is skew-symmetric with respect to $\langle\ ,\ \rangle$ if

$$\langle A(v), w\rangle = -\langle v, A(w)\rangle$$

for all v and w.

(96) As an example we consider vector bundles of small rank equipped with metric connections. If the rank is $= 1$, then the skew-symmetry of the connection matrix shows that every metric connection is flat. As every bundle admits a metric and a compatible connection, we conclude that all rank one bundles admit flat connections.

22 January 2014

(97) We now discuss metric connections on rank 2 bundles. Here the curvature is determined by the closed 2-form Ω_{12} with respect to an orthonormal frame. In the oriented case, this closed form is the same for all oriented orthonormal frames. Therefore, for an oriented rank 2 bundle $E \longrightarrow M$ one defines the **Euler class** $e(E) = -\frac{1}{2\pi}[\Omega_{12}] \in H^2_{dR}(M)$. Changing the orientation of E changes the sign of its Euler class. The Euler class does not depend on the metric connection ∇. In fact, one can show that it does not depend on the metric either, and is therefore a topological invariant of vector bundles.

(98) If E admits a nowhere vanishing section, then $e(E) = 0$. In particular the Euler class of a trivial bundle vanishes.

(99) If M is an oriented surface, then TM is an oriented rank 2 bundle. One of the significant properties of the Euler class is that if M has no boundary then

$$-\frac{1}{2\pi}\int_M \Omega_{12} = \chi(M) = 2 - 2g(M)$$

is the Euler characteristic of M, where $g(M)$ is the genus of the surface. This is the **Gauss–Bonnet theorem**, which we do not prove today.

(100) On a smooth manifold M we now consider connections ∇ on the tangent bundle $TM \to M$. These are sometimes called **affine connections**. In this case the variables X and s in $\nabla_X s$ are on equal footing, as they are both sections of the tangent bundle. This leads to possible symmetries which make no sense in the more general setting of arbitrary vector bundles.

(101) The **torsion** of a connection ∇ on TM is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

for all $X, Y \in \mathcal{X}(M)$.

Lemma 15. *The torsion defines a skew-symmetric map*

$$T\colon \mathcal{X}(M)\times\mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

that is bilinear over $C^\infty(M)$.

A connection ∇ is called **symmetric** if it is torsion-free, i. e. if T vanishes identically[1].

(102) To explain why torsion-freeness is indeed a symmetry condition, we consider the expression of the connection in a local coordinate system $(x_1,\ldots,x_n)$ on M. We write ∂_i for the coordinate vector fields $\frac{\partial}{\partial x_i}$, and use the local frame $\partial_1,\ldots,\partial_n$. Then

$$\nabla\partial_i = \sum_{j=1}^n \omega_{ij}\otimes\partial_j\ ,$$

and using (7) we obtain

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \omega^i_{jk}\partial_k\ ,$$

which is usually written as

$$\nabla_{\partial_i}\partial_j = \sum_{k=1}^n \Gamma^k_{ij}\partial_k$$

in classical notation. Therefore, we define the **Christoffel symbols** of the connection ∇ with respect to the coordinate system $(y_1,\ldots,y_n)$ to be $\Gamma^k_{ij} = \omega^i_{jk}$.

Returning to the definition of torsion, we see that

$$T(\partial_i,\partial_j) = \sum_{k=1}^n(\omega^i_{jk} - \omega^j_{ik})\partial_k = \sum_{k=1}^n(\Gamma^k_{ij} - \Gamma^k_{ji})\partial_k\ .$$

As the torsion is linear over the smooth functions, we obtain the following:

Lemma 16. *An connection ∇ on the tangent bundle is torsion-free if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$ for any local coordinate system.*

[1]Note that requiring the naive symmetry $\nabla_X Y = \nabla_Y X$ for all X and Y leads to a contradiction.

Thus symmetry of the connection really refers to a symmetry of the Christoffel symbols expressing this connection in local coordinates.

(103) If $E \to M$ is a vector bundle with a connection ∇, then the dual bundle $E^* \to M$ carries a well-defined dual connection ∇^* characterized by the identity

$$d\langle s, \alpha\rangle = \langle \nabla s, \alpha\rangle + \langle s, \nabla^* \alpha\rangle$$

for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$. (The brackets here denote the natural pairing between a bundle and its dual bundle, not a metric.)

In the case of a connection on the tangent bundle, the dual connection ∇^* on T^*M gives us the following characterization of torsion-freeness:

Proposition 17. *An connection ∇ on TM is torsion-free if and only if the exterior derivative on one-forms is given by the composition*

$$\Omega^1(M) = \Gamma(T^*M) \xrightarrow{\nabla^*} \Gamma(T^*M \otimes T^*M) \xrightarrow{\wedge} \Gamma(\Lambda^2 T^*M) = \Omega^2(M) \ .$$

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(104) First we prove the above Proposition.

(105) Consider a **Riemannian manifold** M, that is a smooth manifold with a metric on its tangent bundle. In this case it turns out that there is a unique symmetric connection that is at the same time compatible with the given metric:

Proposition 18 (Fundamental Lemma of Riemannian Geometry)**.** *The tangent bundle of a Riemannian manifold admits a unique torsion-free connection compatible with the metric.*

Proof. Given vector fields X, Y and Z, we can use the two requirements, compatibility with the metric $\langle\ ,\ \rangle$ and torsion-freeness, to conclude that the only possible value for $\langle \nabla_X Y, Z\rangle$ is

$$\begin{aligned}(8)\quad \langle \nabla_X Y, Z\rangle = \frac{1}{2}(\langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle + \langle [Z,Y], X\rangle + \\ L_X\langle Y, Z\rangle + L_Y\langle X, Z\rangle - L_Z\langle X, Y\rangle) \ .\end{aligned}$$

This proves uniqueness. To see existence, we use (8) as a definition. As $\langle\ ,\ \rangle$ is nondegenerate, requiring that the equation hold for all Z uniquely defines $\nabla_X Y$. We then check that this ∇ is indeed a connection, is metric-compatible, and torsion-free. ☐

The formula (8) is sometimes called the **Koszul formula**.

(106) The curvature of a Riemannian manifold $(M, \langle\ ,\ \rangle)$ is, by definition, the curvature of its **Levi-Civita connection** ∇ given by the Fundamental Lemma of Riemannian Geometry. We write R for the curvature F^∇ of ∇, and consider this either as

$$\begin{aligned} R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow \mathcal{X}(M) \\ (X, Y, Z) &\longmapsto R(X,Y)Z \ ,\end{aligned}$$

or as

$$\begin{aligned} R\colon \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) &\longrightarrow C^\infty(M) \\ (X, Y, Z, T) &\longmapsto \langle R(X,Y)Z, T\rangle \ .\end{aligned}$$

The notation $R(X, Y)Z$ means that the curvature 2-form is evaluated on X and Y, and the resulting endomorphism of TM is applied to Z. Both of these incarnations of R are called the **Riemann curvature tensor** of $(M, \langle \ , \ \rangle)$; it is a tensor because it is function-linear in all arguments.

(107) The Riemann curvature tensor of a Riemannian manifold $(M, \langle \ , \ \rangle)$ is skew-symmetric in X and Y, and has the following additional symmetries:

Lemma 19. *For all $X, Y, Z, W \in \mathcal{X}(M)$ we have:*
(0) $R(X, Y)Z = -R(Y, X)Z$,
(1) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$,
(2) $\langle R(X, Y)Z, W\rangle = -\langle R(X, Y)W, Z\rangle$,
(3) $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$.

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(108) Let X and Y be two linearly independent tangent vectors in T_pM. Then the expression

$$K(X, Y) = \frac{\langle R(X, Y)Y, X\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} \tag{9}$$

only depends in the two-dimensional subspace $\sigma = span\{X, Y\} \subset T_pM$, and not on the basis X and Y. This is called the **sectional curvature** of σ. It is a high-dimensional analog of the Gaussian curvature of surfaces.

(109) By definition, the sectional curvature is determined by the Riemann tensor. However, the converse is also true:

Lemma 20. *If two Riemannian metrics on M have the same sectional curvatures for all tangent 2-planes, then their curvature tensors R agree.*

The proof is just linear algebra, using Lemma 19 and multiple polarisation, arguing only with the trilinear map

$$R \colon T_pM \times T_pM \times T_pM \longrightarrow T_pM$$

at a point.

(110) The proof the previous Lemma can easily be adapted to prove the following characterization of spaces with constant sectional curvature:

Lemma 21. *A Riemannian manifold $(M, \langle \ , \ \rangle)$ has sectional curvature equal to a fixed real number $K_0 \in \mathbb{R}$ for all two-planes $\sigma \subset TM$ if and only if the following identity holds for all X, Y, Z and $T \in \mathcal{X}(M)$:*

$$\langle R(X, Y)Z, T\rangle = -K_0(\langle X, Z\rangle\langle Y, T\rangle - \langle Y, Z\rangle\langle X, T\rangle) \ .$$

The proof is again just linear algebra at a single point.

(111) To obtain some concrete calculations of the curvature of a Riemannian manifold, we now consider the following soecial situation: $M^n \subset \mathbb{R}^{n+1}$ is a smooth submanifold of codimension one in Euclidean space, and the Riemannian metric on M is the one induced by the standard scalar product on $\mathbb{R}^{n+1}$. If M is oriented, or we work locally, then there is

a unique unit normal vector $n(p)$ at every point $p \in M$ such that the orientation of T_pM together with $n(p)$ gives the positive orientation of $\mathbb{R}^{n+1}$. The smooth map

$$\begin{aligned} G\colon M &\longrightarrow S^n \\ p &\longmapsto n(p) \end{aligned}$$

is called the Gauss map of M.

Lemma 22. *The tangent spaces T_pM and $T_{G(p)}S^n$ are the same subspace of $\mathbb{R}^{n+1}$. Under this identification, the derivative D_pG is given by*

$$\begin{aligned} L\colon T_pM &\longrightarrow T_pM \\ X_p &\longmapsto (\tilde{\nabla}_X n)(p) \ , \end{aligned}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\mathbb{R}^{n+1}$ equipped with the constant scalar product.

The map L is called the **Weingarten map** of the hypersurface M. It is self-adjoint with respect to the metric on M.

Now for X and Y vector fields on M, we can choose local extensions $\tilde{X}$ and $\tilde{Y}$ on $\mathbb{R}^{n+1}$ and define

$$\nabla_X Y = \pi(\tilde{\nabla}_{\tilde{X}} \tilde{Y}) \ ,$$

where π is the orthogonal projection $\mathbb{R}^{n+1} \longrightarrow TM$ with kernel spanned by the unit normal n. We leave it as an exercise to show that this is in fact the Levi-Civita connection of the metric on M.

With this relationship between ∇ and $\tilde{\nabla}$ in hand, we can compare their curvatures $R(X,Y)Z$ and $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}$. On $\mathbb{R}^{n+1}$ the directional covariant derivatives commute, which means that the curvature vanishes. The equation $\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z} = 0$ can be split into two equations asserting the vanishing of the component tangent to M, and the vanishing of the normal component. These two equations give the following:

Proposition 23. *For a smooth hypersurface in Euclidean space equipped with the induced metric we have*

$$R(X,Y)Z = \langle L(Y), Z\rangle L(X) - \langle L(X), Z\rangle L(Y) \quad \textit{(the Gauss equation)}$$

and

$$\nabla_X(L(Y)) - \nabla_Y(L(X)) = L([X,Y]) \quad \textit{(the Codazzi–Mainardi equation).}$$

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